Lower Bounds on the Depth of Integral ReLU Neural Networks via Lattice Polytopes

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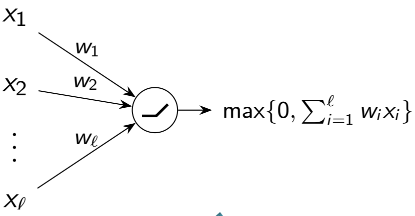
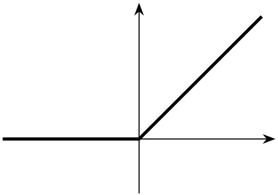
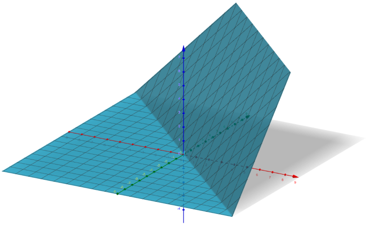
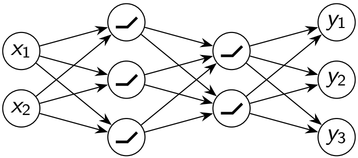
ReLU Neural Networks

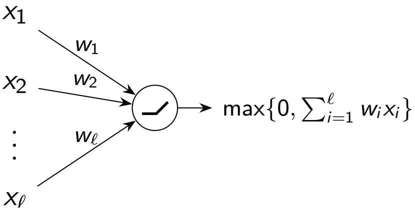


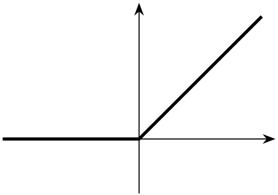


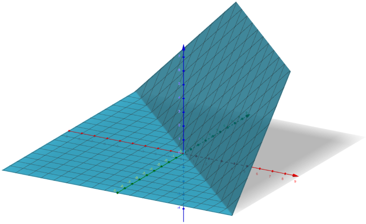


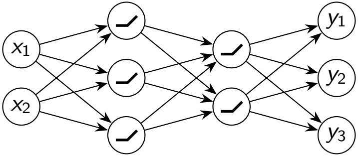


What is the class of functions computable by
ReLU Neural Networks
with a certain number of layers?



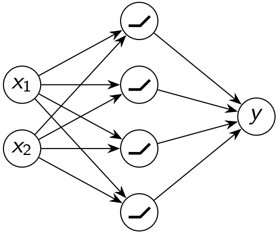






Universal approximation theorems:

One hidden layer enough to **approximate** any continuous function.



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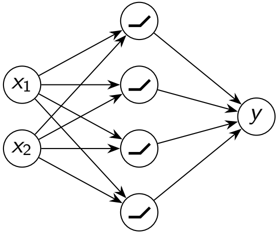
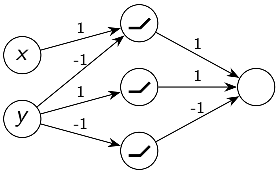


What about **exact** representability?

Example: Computing the Maximum of Two Numbers

$$\max\{x, y\} = \max\{x - y, 0\} + y$$



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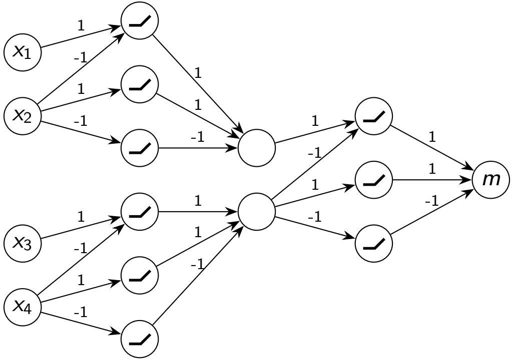

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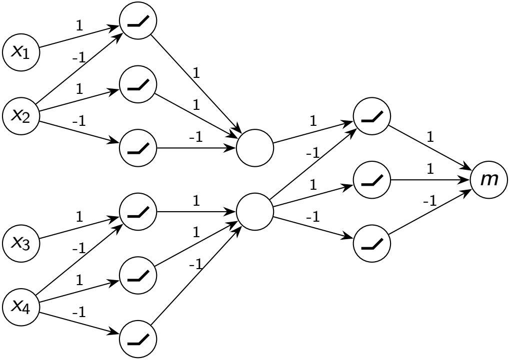

- Inductively: Max of n numbers with $\lceil \log_2(n) \rceil$ hidden layers.

More Generally ...

Theorem (Arora, Basu, Mianjy, Mukherjee (2018))

Every continuous, piecewise linear function $f : \mathbb{R}^n \to \mathbb{R}$ *can be represented by a ReLU NN with* $\lceil \log_2(n+1) \rceil$ *hidden layers.*

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- Is logarithmic depth best possible?

Conjecture (Hertrich, Basu, Di Summa, Skutella (2021))

Yes, there are functions which need $\lceil \log_2(n+1) \rceil$ *hidden layers!*

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This is equivalent to:

Conjecture (Hertrich, Basu, Di Summa, Skutella (2021))

$\max\{0, x_1, \ldots, x_{2^k}\}$ *cannot be represented with* k *hidden layers.*

Results

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- **We show:**
  Conjecture holds for all k if network has only integer weights!

Proof Techniques

- Use tropical geometry to represent NNs as lattice polytopes. (Compare Zhang, Naitzat, Lim (2018))

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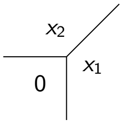

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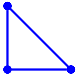

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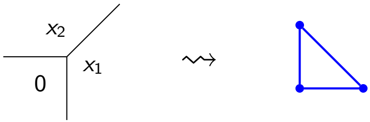
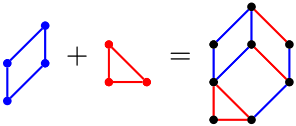


- Subdivide polytopes “layer by layer” into “easier” polytopes.



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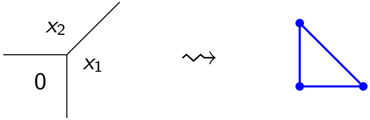

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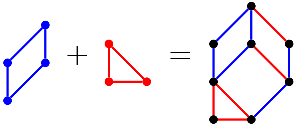

- Separate via parity of the normalized volume.

Outlook

To prove general conjecture ...

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Thanks for watching!