Faster Binary Embeddings for Preserving Euclidean Distances

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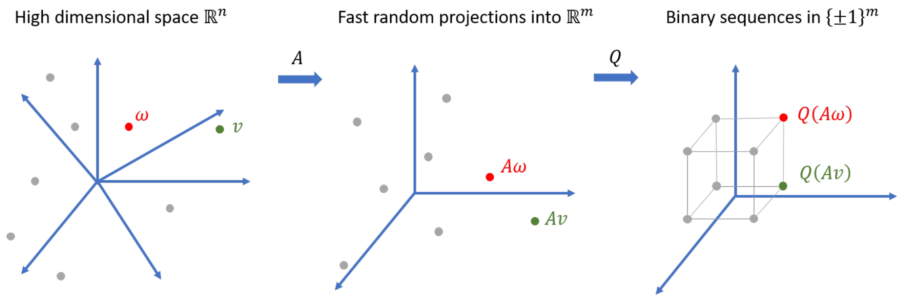
Wednesday 24th March, 2021

Framework



Euclidean distance recovery: $d(Q(A\omega), Q(Av)) \approx \| \omega - v \|_2$

- Our method is

$$q_x := Q(Ax)$$

where Q is a stable $\Sigma\Delta$ quantization scheme, $A \in \mathbb{R}^{m \times n}$ is a sparse Gaussian matrix and $x \in \mathbb{R}^n$ is well-spread, i.e., those that are not sparse.

Sigma-Delta Quantization Schemes Q

For simplicity, we focus on the quantizer $Q : \mathbb{R}^m \to \{1, -1\}^m$. Specifically, $q = Q(x)$ such that for $i = 1, ..., m$

$$\begin{cases} u_0 = 0, \\ q_i = \text{sign}(x_i + u_{i-1}), \\ u_i = u_{i-1} + x_i - q_i. \end{cases} \tag{1}$$

Algorithm 1: Fast Binary Embedding for Finite $\mathcal{T}$

Input: $\mathcal{T} = \{x^{(j)}\}_{j=1}^k \subseteq B_2^n(\kappa)$ ▷ Data points in ℓ_2 ball
Generate $A \in \mathbb{R}^{m \times n}$ ▷ Sparse Gaussian matrix A
for $j \leftarrow 1$ **to** k **do**
  $z^{(j)} \leftarrow Ax^{(j)}$
  $q^{(j)} = Q(z^{(j)})$ ▷ Stable $\Sigma\Delta$ quantizer Q as in (1)
Output: Binary sequences $\mathcal{B} = \{q^{(j)}\}_{j=1}^k \subseteq \{-1, 1\}^m$

ℓ_2 Distance Recovery

Algorithm 2: ℓ_2 Norm Distance Recovery

Input: $q^{(i)}, q^{(j)} \in \mathcal{B}$ ▷ Binary sequences produced by Algorithm 1
$y^{(i)} \leftarrow \widetilde{V} q^{(i)}$ ▷ Condense the components of q
$y^{(j)} \leftarrow \widetilde{V} q^{(j)}$
Output: $\|y^{(i)} - y^{(j)}\|_1$ ▷ Approximation of $\|x^{(i)} - x^{(j)}\|_2$

Definition (Condensation operator)

Let p, r, λ be fixed positive integers such that $\lambda = r\widetilde{\lambda} - r + 1$ for some integer $\widetilde{\lambda}$. Let $m = \lambda p$ and v be a row vector in $\mathbb{R}^\lambda$ whose entry v_j is the j-th coefficient of the polynomial $(1 + z + \ldots + z^{\widetilde{\lambda}-1})^r$. Define the normalized condensation operator $\widetilde{V} \in \mathbb{R}^{p \times m}$ by

$$\widetilde{V} = \frac{\sqrt{\pi/2}}{p\|v\|_2} \begin{bmatrix} v & & \\ & \ddots & \\ & & v \end{bmatrix}.$$

Main Result

Theorem (Main result)

Let $\mathcal{T} \subseteq \mathbb{R}^n$ be a finite, appropriately scaled set with elements satisfying $\|x\|_\infty = O(n^{-1/2}\|x\|_2)$. If $m \gtrsim p := \Omega(\epsilon^{-2}\log(|\mathcal{T}|^2/\delta))$ and $r \geq 1$ is the integer order of Q, then with probability $1 - 2\delta$ on the draw of the sparse Gaussian matrix A, the following holds uniformly over all x, y in $\mathcal{T}$: Embedding x, y into $\{-1, 1\}^m$ using Algorithm 1, and estimating the associated distance between them using Algorithm 2 yields the error bound

$$\left| \|\widetilde{V}(q_x - q_y)\|_1 - \|x - y\|_2 \right| \leq c\left(\frac{m}{p}\right)^{-r+1/2} + \epsilon\|x - y\|_2.$$

- The assumption that $\|x\|_\infty = O(n^{-1/2}\|x\|_2)$ is reasonable.
- The latter part in error bound is essentially proportional to $p^{-1/2}$.

Comparisons with Baselines

Method	Time	Space	Storage	Query Time
Toeplitz [1]	$O(n \log n)$	$O(n)$	$O(m)$	$O(m)$
Bilinear [2]	$O(n\sqrt{m})$	$O(\sqrt{mn})$	$O(m)$	$O(m)$
Circulant [3]	$O(n \log n)$	$O(n)$	$O(m)$	$O(m)$
BOE or PCE [4]	$O(n \log n)$	$O(n)$	$O(p \log_2 \lambda)$	$O(p \log_2^2 \lambda)$
Our Algorithm [5]	$O(m)$	$O(m)$	$O(p \log_2 \lambda)$	$O(p \log_2 \lambda)$

- "Time" is the time needed to embed a data point;
- "Space" is the space needed to store the embedding matrix;
- "Storage" contains the memory usage to store each encoded sequence;
- "Query time" is the time complexity of pairwise distance estimation.

Experiments on different Image Datasets

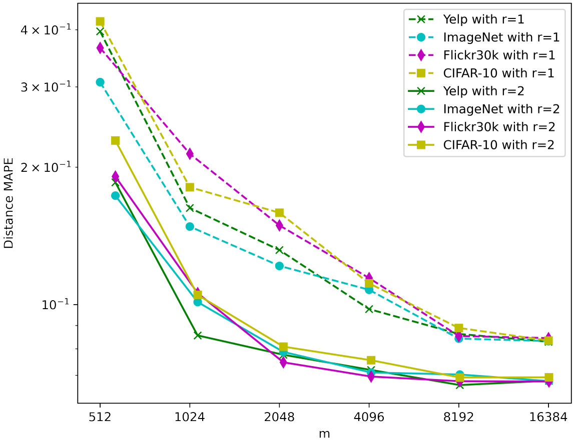


Figure: Plot of ℓ_2 distance reconstruction error on four datasets

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Thank You