Leveraging augmented-Lagrangian techniques for differentiating over infeasible quadratic programs in machine learning

Antoine Bambade[1,2]

[1]*Inria and ENS Paris : Willow and Sierra teams*
[2]*École des Ponts Paris Tech*

Joint work with Fabian Schramm, Adrien Taylor, Justin Carpentier

Quadratic programming layer pipeline

More recent literature considers differentiable optimization problems as layers.

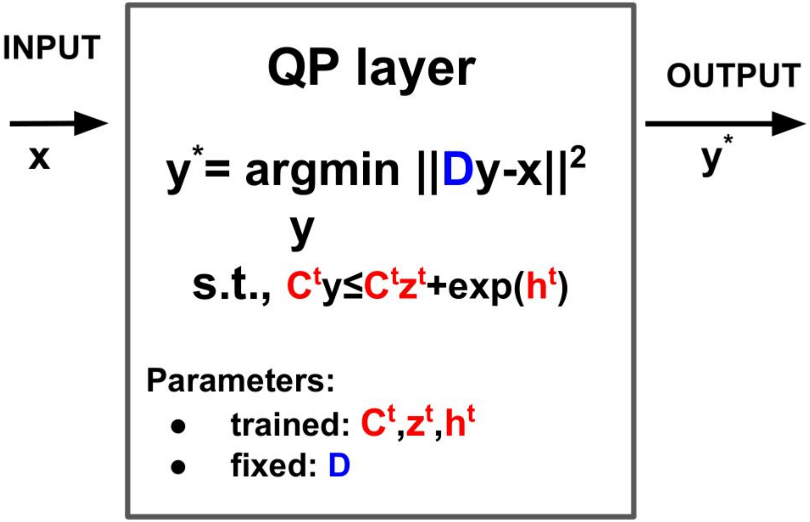


Figure: Example of a Quadratic Programming Layer (with D nonsingular)

QP layers in machine learning

Convex QP layers performs better than a ConvNet for solving Sudokus.

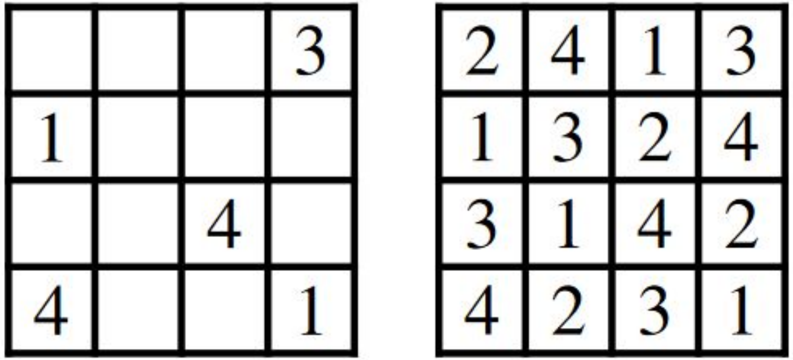


Figure: Example of Sudoku.

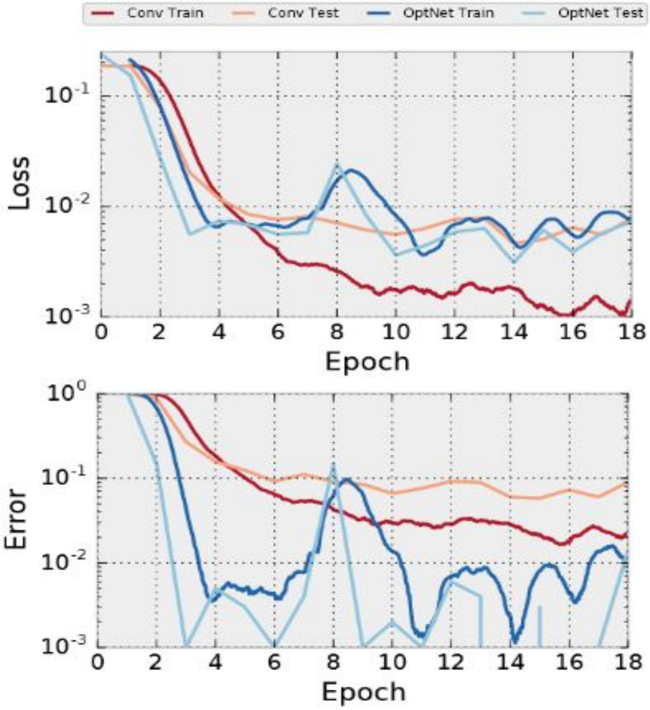


Figure: Training and test plots[1].

[1]B. Amos, Z. Kolter (2021)

QP layers cons: limited trainable architecture

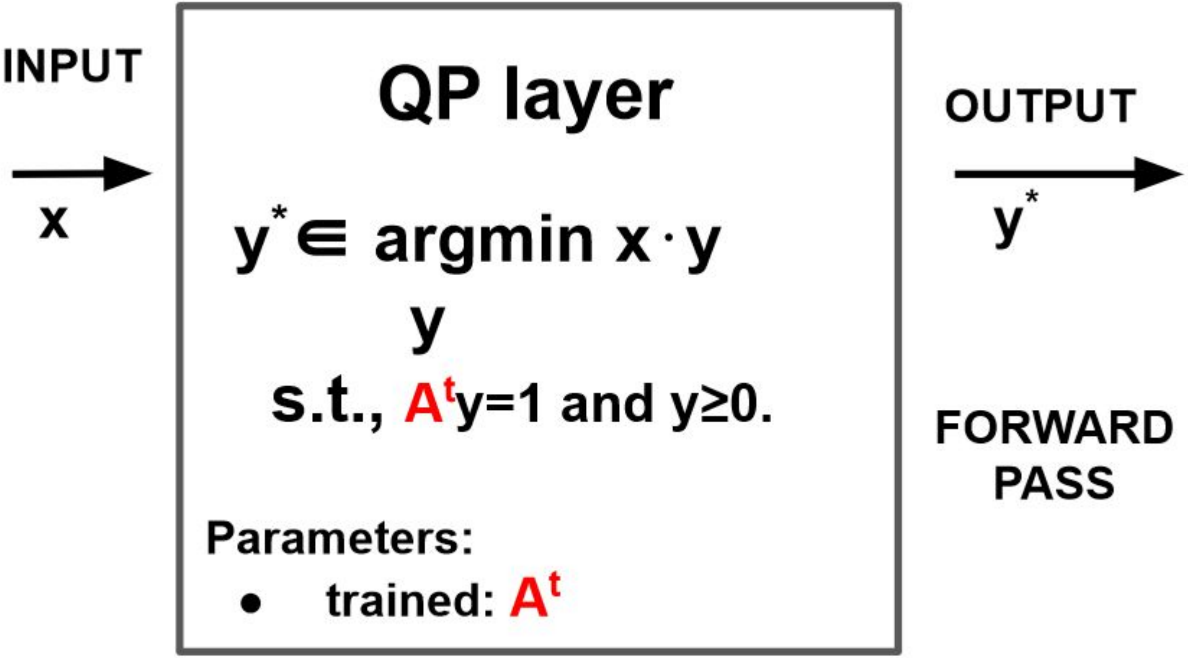
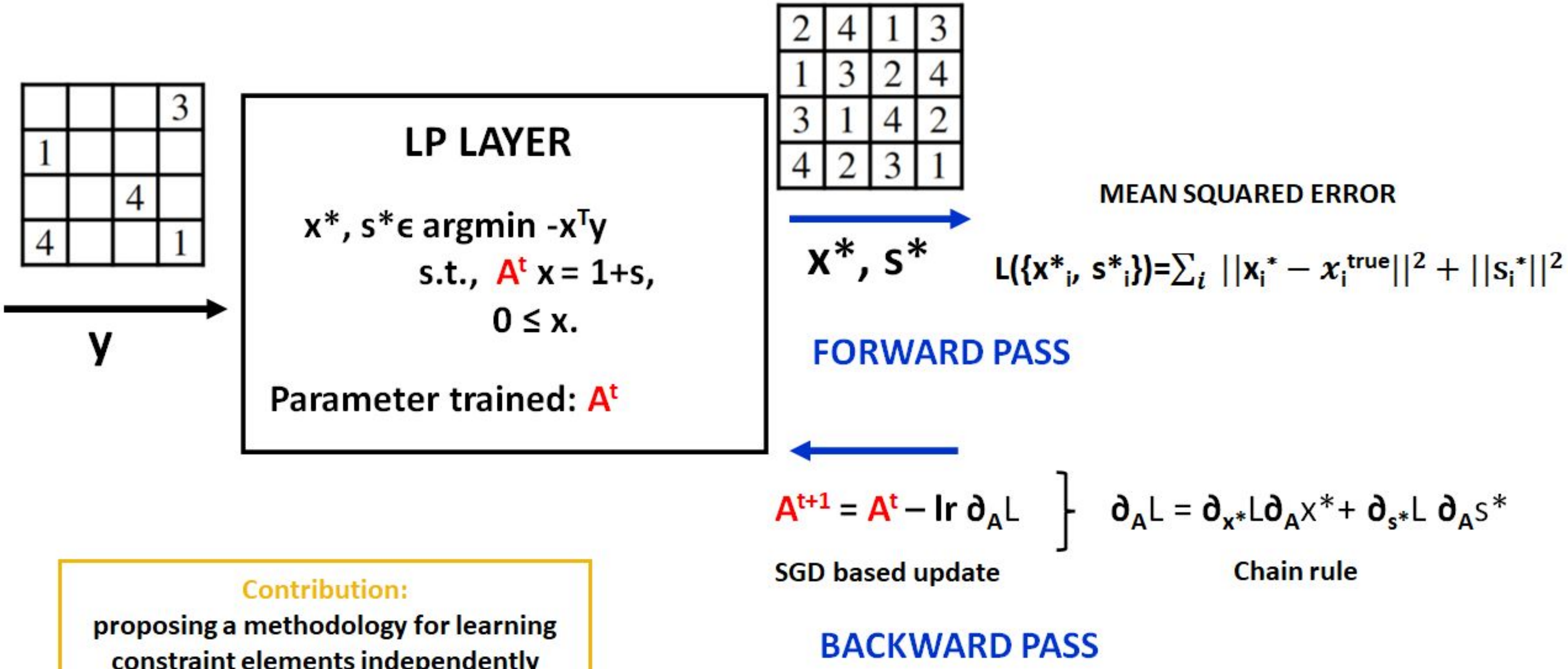


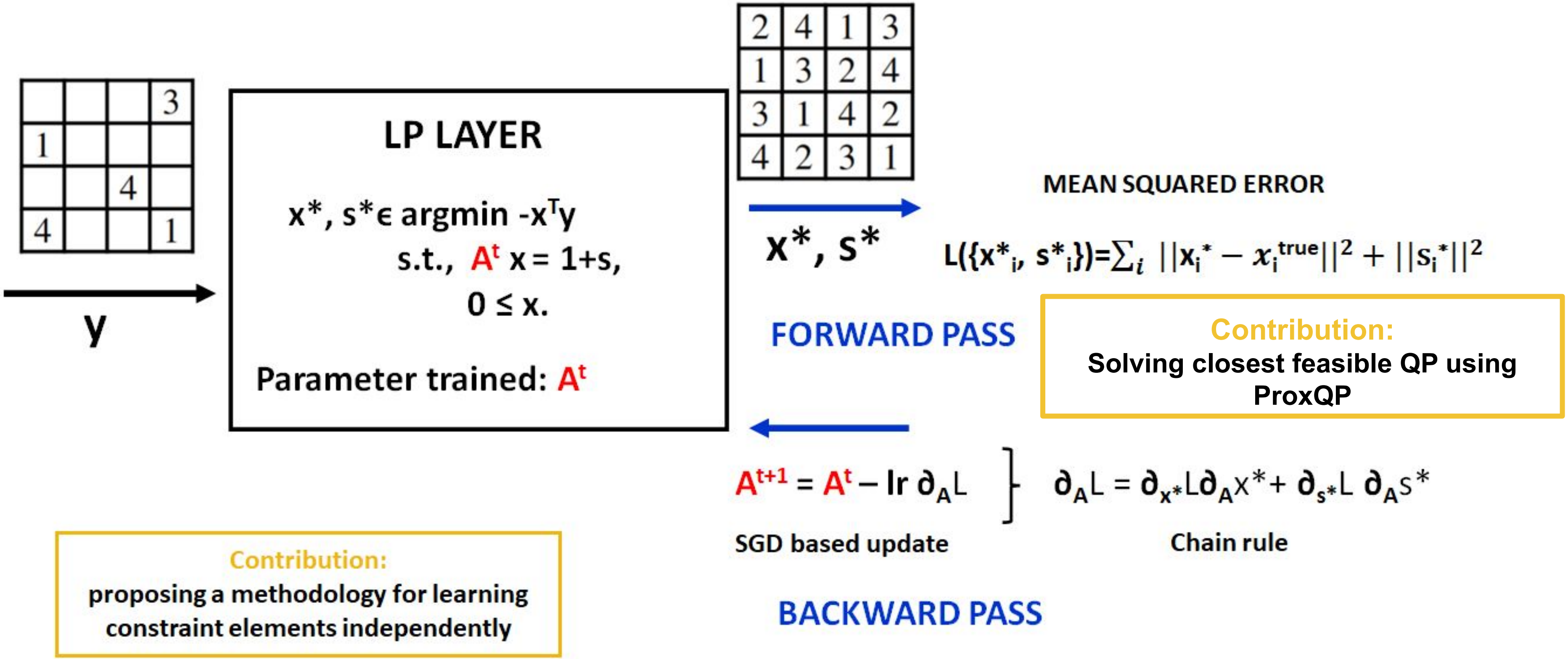
Figure: a LP layer. Nothing guarantees during training that the vector of 1 lies in the range space of A^t.

Solution outline: ideal pipeline



Contribution:
proposing a methodology for learning constraint elements independently

Solution outline: ideal pipeline



A. Chiche, J-C. Gilbert (2016)

Software contribution

ProxSuite

THE ADVANCED PROXIMAL OPTIMIZATION TOOLBOX

License BSD 2-Clause | docs online | CI - Linux/OSX/Windows - Cond passing | pypi package 0.6.1 | Anaconda.org 0.6.1

- ✓ **fast**: C++ implementation, with homemade linear Cholesky solver
- ✓ **scalable:** various backends for dense, sparse and matrix-free optimization
- ✓ **easy-to-use:** API closed to OSQP, Python and Julia bindings
- ✓ **open-source:** BSD-license, easily installable

Conda | Files | Labels | Badges

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</> Development: https://github.com/simple-robotics/proxsuite

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The backward pass: differentiating closest QP solutions

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A classical technique: **the Implicit Function Theorem.**

The backward pass: differentiating closest QP solutions

$$
\begin{aligned}
s^\star(\theta) = & \arg \min_{s \in \mathbb{R}^{n_i}} \ \frac{1}{2}\|s\|_2^2 \\
& \text{s.t. } x^\star(\theta), z^\star(\theta) \in \arg \min_{x \in \mathbb{R}^n} \max_{z \in \mathbb{R}_+^{n_i}} L(x, z, s;\ \theta),
\end{aligned}
$$

with $L(x, z, s;\ \theta) \triangleq f(x;\ \theta) + z^\top (C(\theta)x - u(\theta) - s)$.

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The backward pass: differentiating closest QP solutions

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A classical technique: **the Implicit Function Theorem.**

$$G(x, z, t;\ \theta) \triangleq \begin{bmatrix} \nabla_x f(x;\ \theta) + C(\theta)^\top z \\ C(\theta)x - u(\theta) - t \\ [[t]_- + z]_+ - z \\ C(\theta)^\top [t]_+ \end{bmatrix}$$

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The map is **path-differentiable**.

E. Pauwels et al. (2019)

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Contribution:
Extend the technique for the closest feasible QP solutions.

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Contribution:
Efficient algorithms to solve these problems.

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$$

Contribution:
QPLayer: A full differentiable pipeline in C++ connected with PyTorch.

A classical technique: **the Implicit Function Theorem.**

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Extend the technique for the closest feasible QP solutions.

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Numerical benchmark: back to the Sodoku problem.

Convex QP layers performs better than a ConvNet for solving Sudokus.

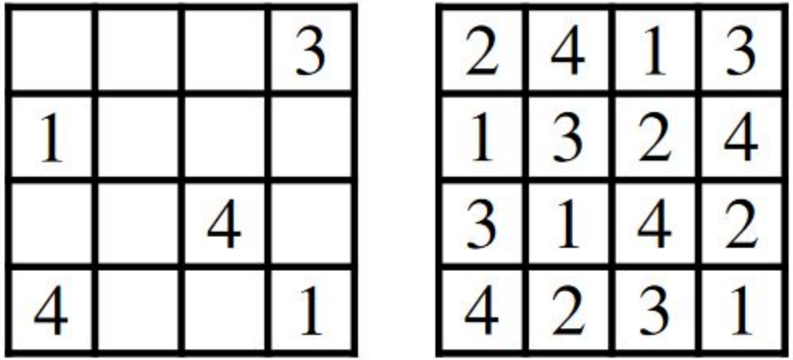


Figure: Example of Sudoku.

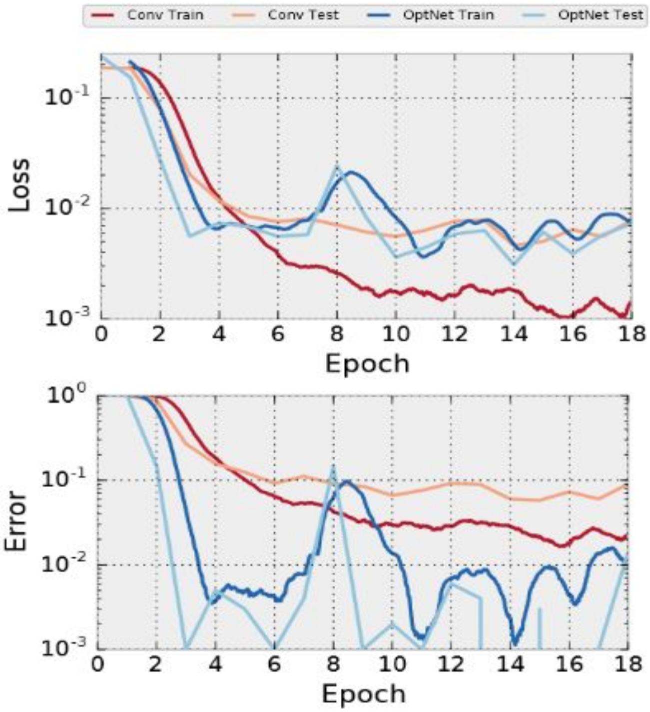
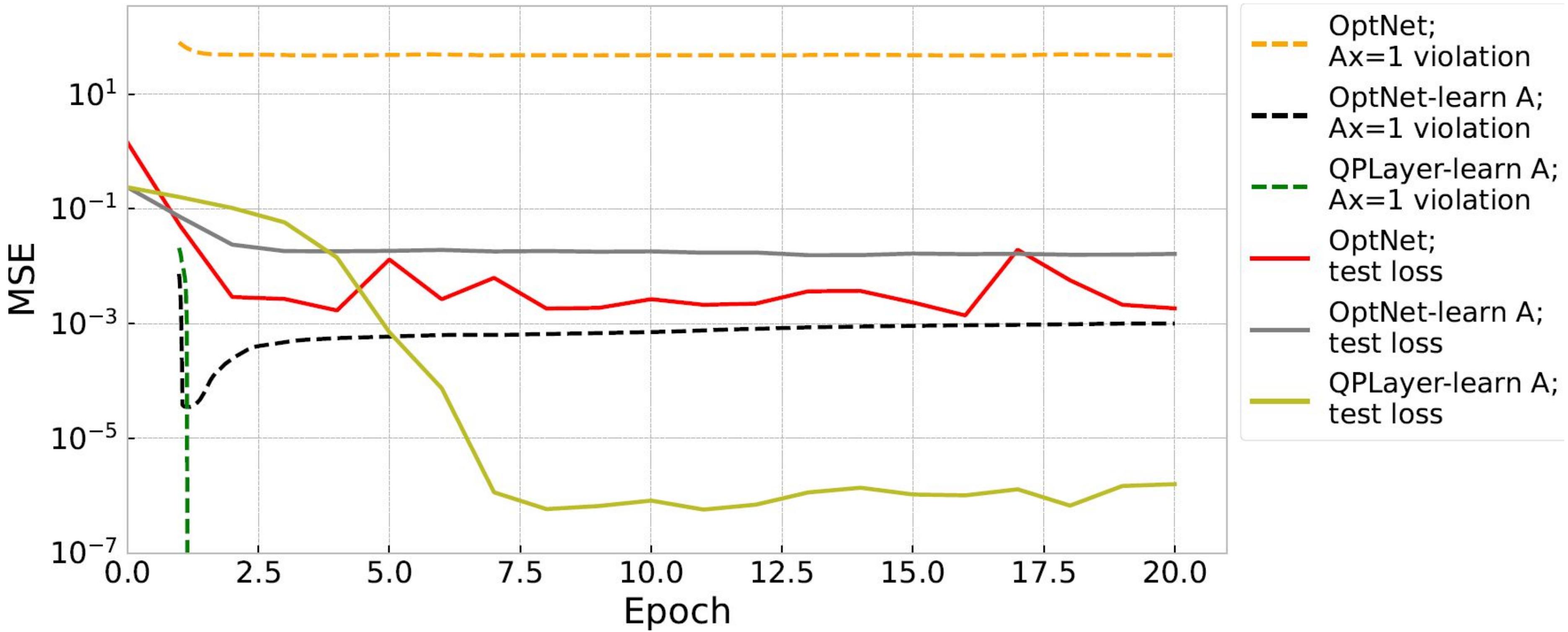


Figure: Training and test plots[1].

[1]B. Amos, Z. Kolter (2021)

Loss comparison



- **Methodology for learning new QP layers**
  - IFT for closest feasible QPs
  - Extended conservative Jacobians
- **QPlayer: open-source differentiable pipeline**
  - Use Augmented-Lagrangian techniques
  - Connected with PyTorch

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