# Weaker MVI Condition

## Extragradient Methods with Multi-Step Exploration

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# Min-max optimization

## Problem setting

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    - Weak MVI allows $\rho < 0$

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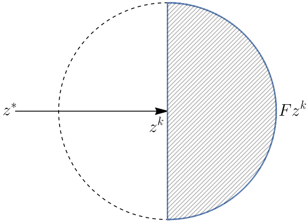


MVI: negative gradient points towards zero

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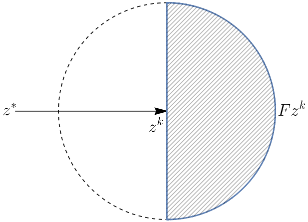


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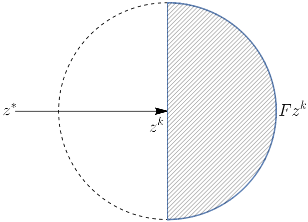


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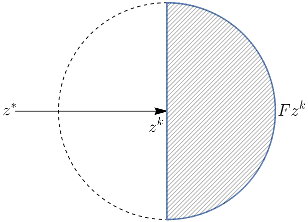
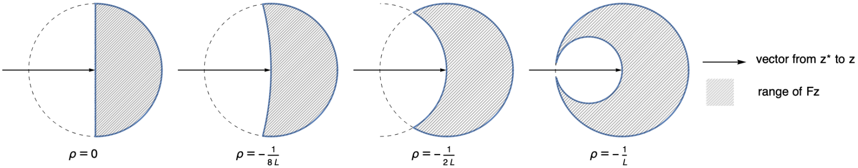
- star: $\boldsymbol{v} = \boldsymbol{z}^*$, hypo: $\rho < 0$




MVI: negative gradient points towards zero

# Weak MVI Condition: Intuitions

- Weak MVI: negative gradient might point away from zero




weak MVI with different parameters

# Existing Works

- Extragradient (MVI)
  - same stepsizes

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- EG+/AdaptiveEG+ (weak MVI, $\rho > -1/2L$)
  - larger extrapolation stepsize $\gamma_k$

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# Our Contributions

- First known first-order algorithm that converges for weak MVI problems with $\rho < -1/2L$.
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- Adaptive method effectively resolves problems with limit cycles.

# Multi-step Extragradient

## 2-step Extragradient

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\begin{aligned}
\boldsymbol{z}_1^k &= \boldsymbol{z}^k - \gamma_{k,1} F \boldsymbol{z}^k \\
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Take stepsizes $\gamma_{k,1} = {}^{\delta_1}/_L$, $\gamma_{k,2} = {}^{\delta_2}/_L$

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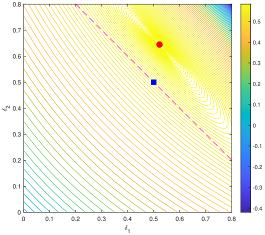


Relation between the upper bound of $-\rho L$ and $\delta_1$, $\delta_2$

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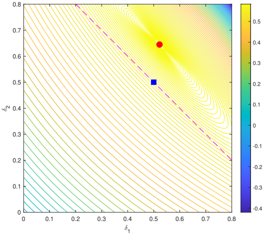


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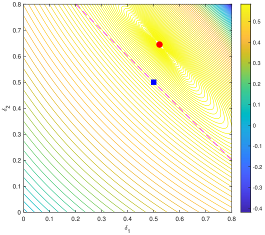
- $\delta_1 = \delta_2 = {}^1/_2$, $\rho > -{}^5/_{9L}$
- $\delta_1 \approx 0.52212$, $\delta_2 \approx 0.644793$, $\rho > -{}^{0.5834}/_L$

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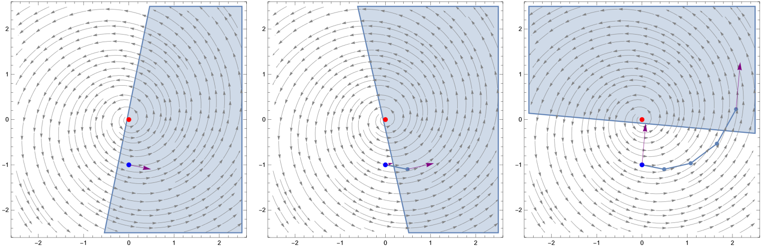
- Convergence guarantee for $\rho > -{}^{(1-1/e)}/_{L} \approx -{}^{0.632}/_{L}$ with invariant stepsizes $\gamma_{k,i} = {}^{1}/_{nL}, i \in [n]$ and large enough $n$

# Introducing Adaptive Exploration

- Perspective of projection



(a) GDA (b) AdaptiveEG+ (c) MDEG

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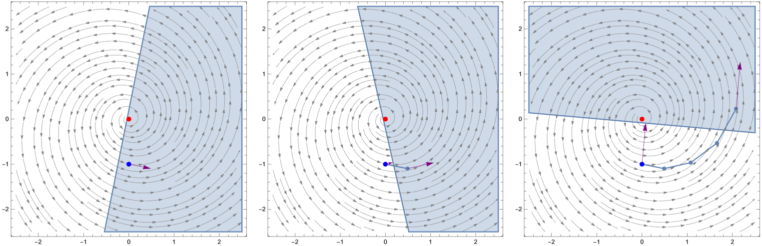


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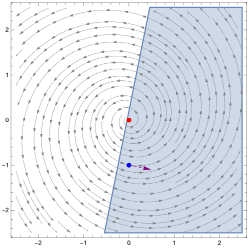


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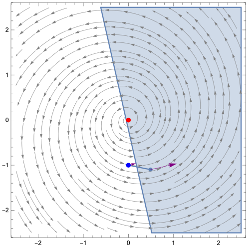
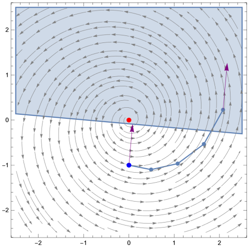



(b) AdaptiveEG+

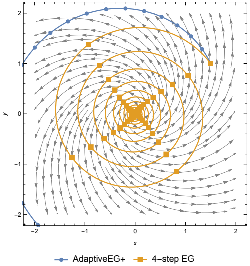



(c) MDEG

- MDEG: Max Distance Extragradient
- Sub-iteration stops when projection distance decreases

# Examples

- *n*-step EG converges for $\rho < -1/2L$.

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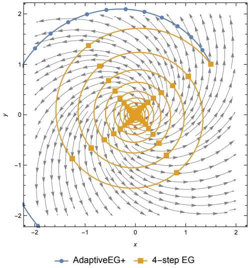


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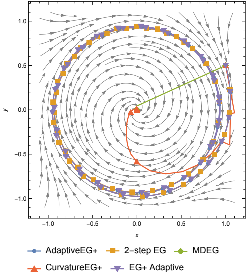


(b) $\rho L \approx -0.885521$

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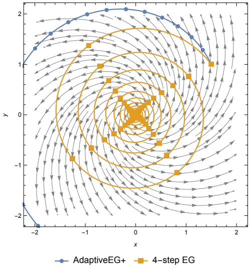
- *n*-step EG converges for $\rho < -1/2L$.
- MDEG converges break out of the limit cycle and converges to the stationary point.
- MDEG bypasses the highly nonmonotonic regions with local $\rho L \approx -3.04076$ and converges to the stationary point.

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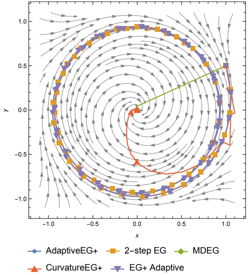


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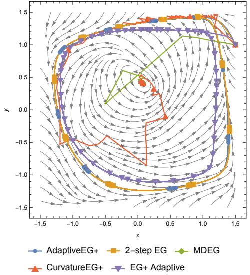


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*Thank you!*