A Fast and Provable Algorithm for Sparse Phase Retrieval

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Sparse phase retrieval

- The phase retrieval problem is to reconstruct an n-dimensional signal $\boldsymbol{x}^\natural$ using its magnitude-only measurements:

$$y_i = |\langle \boldsymbol{a}_i, \boldsymbol{x}^\natural \rangle|^2, \quad i = 1, 2, \cdots, m. \tag{1}$$

$\{y_i\}_{i=1}^m$: phaseless measurements; $\{\boldsymbol{a}_i\}_{i=1}^m$: sensing vectors; $\boldsymbol{x}^\natural$: target signal to be recovered.

- The sparse phase retrieval problem can be expressed as

$$\text{Find } \ \boldsymbol{x}, \quad \text{subject to } \ |\langle \boldsymbol{a}_i, \boldsymbol{x} \rangle|^2 = y_i \ \ \forall\, i = 1, \ldots, m, \ \text{ and } \ \|\boldsymbol{x}\|_0 \le s. \tag{2}$$

- Phase retrieval arises in many applications in scientific imaging, such as diffraction imaging, astronomical imaging, X-ray crystallography, optics, etc.

Overview of state-of-the-art algorithms

Methods	Per-iteration computation	Iteration complexity	Loss function
ThWF [1]	$O(n^2 \log n)$	$O(\log(1/\epsilon))$	$f_I(\boldsymbol{x})$
SPARTA [2]	$O(ns^2 \log n)$	$O(\log(1/\epsilon))$	$f_A(\boldsymbol{x})$
CoPRAM [3]	$O(ns^2 \log n)$	$O(\log(1/\epsilon))$	$f_A(\boldsymbol{x})$
HTP [4]	$O((n + s^2)s^2 \log n)$	$O(\log(\log(n^{s^2})) + \log(\lVert\boldsymbol{x}^\natural\rVert/x^\natural_{\min}))$	$f_A(\boldsymbol{x})$
Proposed	$O((n + s^2)s^2 \log n)$	$O(\log(\log(1/\epsilon)) + \log(\lVert\boldsymbol{x}^\natural\rVert/x^\natural_{\min}))$	$f_I(\boldsymbol{x})$, $f_A(\boldsymbol{x})$

- n: signal dimension; s: sparsity; $x^\natural_{\min}$: the smallest nonzero entry in magnitude of $\boldsymbol{x}^\natural$.
- Our algorithm achieves the lowest iteration complexity, while maintaining the same per-iteration cost as first-order methods, assuming $s = O(\sqrt{n})$ (otherwise, the complexity $\Omega(s^2 \log n)$ for sparse phase retrieval would reduce to that of general methods).

Proposed algorithm

- Two prevalent loss functions: Intensity-based empirical loss

$$f_I(\boldsymbol{x}) := \frac{1}{4m} \sum_{i=1}^{m} \left(|\langle \boldsymbol{a}_i, \boldsymbol{x} \rangle|^2 - y_i \right)^2, \tag{3}$$

- Amplitude-based empirical loss

$$f_A(\boldsymbol{x}) := \frac{1}{2m} \sum_{i=1}^{m} \left(|\langle \boldsymbol{a}_i, \boldsymbol{x} \rangle| - \sqrt{y_i} \right)^2. \tag{4}$$

- The intensity-based loss $f_I(\boldsymbol{x})$ is smooth, while the amplitude-based loss $f_A(\boldsymbol{x})$ is non-smooth because of the modulus.

Proposed algorithm

Step 1: Identify *free* and *fixed* variables

- We identify *free* variables using amplitude-based loss $f_A(\boldsymbol{x})$:

$$\mathcal{S}_{k+1} = \operatorname{supp}\left(\mathcal{H}_s(\boldsymbol{x}^k - \eta \nabla f_A(\boldsymbol{x}^k))\right),$$

  $\mathcal{H}_s$: s-sparse hard-thresholding operator.

- The set of *fixed* variables is the complement of $\mathcal{S}_{k+1}$.
- Only *free* variables are updated using the (approximate) Newton direction, while the *fixed* variables are directly set to zero.

Step 2: Compute search direction

- We update *free* variables based on intensity-based loss f_I:

$$\boldsymbol{x}^{k+1}_{\mathcal{S}_{k+1}} = \boldsymbol{x}^{k}_{\mathcal{S}_{k+1}} - \boldsymbol{p}^{k}_{\mathcal{S}_{k+1}}, \tag{5}$$

$\boldsymbol{p}^{k}_{\mathcal{S}_{k+1}}$: approximate Newton direction over $\mathcal{S}_{k+1}$, calculated by

$$\boldsymbol{H}^{k}_{\mathcal{S}_{k+1},\mathcal{S}_{k+1}} \boldsymbol{p}^{k}_{\mathcal{S}_{k+1}} = -\boldsymbol{H}^{k}_{\mathcal{S}_{k+1},J_{k+1}} \boldsymbol{x}^{k}_{J_{k+1}} + \boldsymbol{g}^{k}_{\mathcal{S}_{k+1}}. \tag{6}$$

$\boldsymbol{g}^{k}_{\mathcal{S}_{k+1}} = \left[\nabla f_I(\boldsymbol{x}^k)\right]_{\mathcal{S}_{k+1}}$, and $\boldsymbol{H}^{k}_{\mathcal{S}_{k+1},\mathcal{S}_{k+1}} = \left[\nabla^2 f_I(\boldsymbol{x}^k)\right]_{\mathcal{S}_{k+1},\mathcal{S}_{k+1}}$.

- Our algorithm exploits second-order information when computing search direction, leading to a faster convergence than projected gradient method.

Proposed algorithm

Algorithm 1: Proposed algorithm

Input: Data $\{\boldsymbol{a}_i, y_i\}_{i=1}^m$, sparsity s, initial estimate $\boldsymbol{x}^0$, and stepsize η.

for $k = 1, 2, \dots$ **do**

Identify the set of *free* variables $\mathcal{S}_{k+1} = \operatorname{supp}(\mathcal{H}_s(\boldsymbol{x}^k - \eta \nabla f_A(\boldsymbol{x}^k)))$;

Compute the search direction $\boldsymbol{p}^k_{\mathcal{S}_{k+1}}$ over $\mathcal{S}_{k+1}$ by solving (6);

Update $\boldsymbol{x}^{k+1}$:

$$\boldsymbol{x}^{k+1}_{\mathcal{S}_{k+1}} = \boldsymbol{x}^k_{\mathcal{S}_{k+1}} - \boldsymbol{p}^k_{\mathcal{S}_{k+1}}, \quad \text{and} \quad \boldsymbol{x}^{k+1}_{\mathcal{S}^c_{k+1}} = \mathbf{0}.$$

end

Output: $\boldsymbol{x}^{k+1}$.

Theoretical analysis

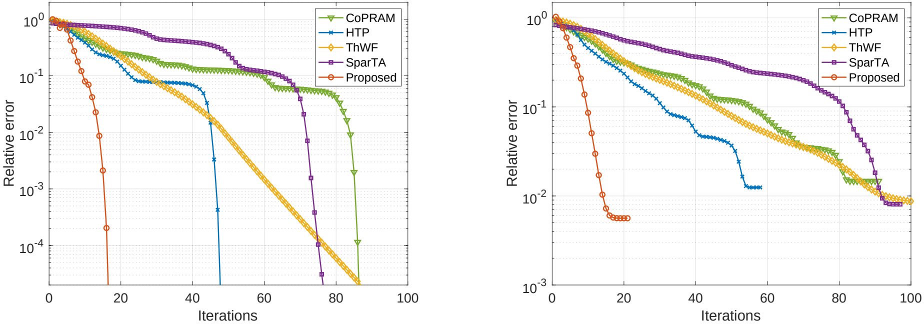
Theorem

If the number of measurements $m \geq c_1 s^2 \log n$*, then with probability at least* $1-(c_2\bar{k}+c_3)m^{-1}$*, the sequence* $\{\boldsymbol{x}^k\}_{k\geq 1}$ *generated by Algorithm 1 converges to the ground truth* $\boldsymbol{x}^\natural$ *at a quadratic rate after at most* $O\big(\log(\|\boldsymbol{x}^\natural\|/x^\natural_{\min})\big)$ *iterations, i.e.,*

$$\operatorname{dist}(\boldsymbol{x}^{k+1},\boldsymbol{x}^\natural) \leq \rho \cdot \operatorname{dist}^2(\boldsymbol{x}^{k},\boldsymbol{x}^\natural), \quad \forall\, k \geq \bar{k},$$

where $\bar{k} \leq c_4 \log\left(\|\boldsymbol{x}^\natural\|/x^\natural_{\min}\right) + c_5$*, and* $x^\natural_{\min}$ *is the smallest nonzero entry in magnitude of* $\boldsymbol{x}^\natural$*.*

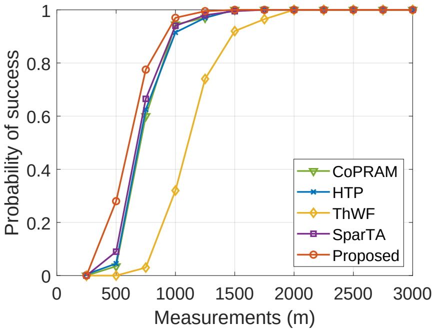
Numerical results



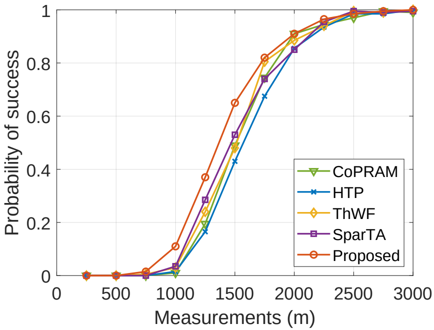
(a) Noise free (b) Noisy

Relative error versus iterations, with signal dimension $n = 5000$ and sample size $m = 3000$.

Numerical results



(a) $s = 25$



(b) $s = 50$

Phase transition for signals of dimension $n = 3000$ with sparsity levels $s = 25$ and 50.

References I

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[2] G. Wang, L. Zhang, G. B. Giannakis, M. Akçakaya, and J. Chen, "Sparse phase retrieval via truncated amplitude flow," *IEEE Transactions on Signal Processing*, vol. 66, no. 2, pp. 479–491, 2017.

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Thank you!