Approximating Family of Steep Traveling Wave Solutions to Fisher's Equation with PINNs

Franz M. Rohrhofer, Stefan Posch, Clemens Gößnitzer, Bernhard C. Geiger

Introduction

Fisher's Equation & Traveling Waves

Fisher's equation, a simple yet profound **reaction-diffusion system**, was introduced by (Fisher, 1937) and reads

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \rho u(1-u), \tag{1}$$

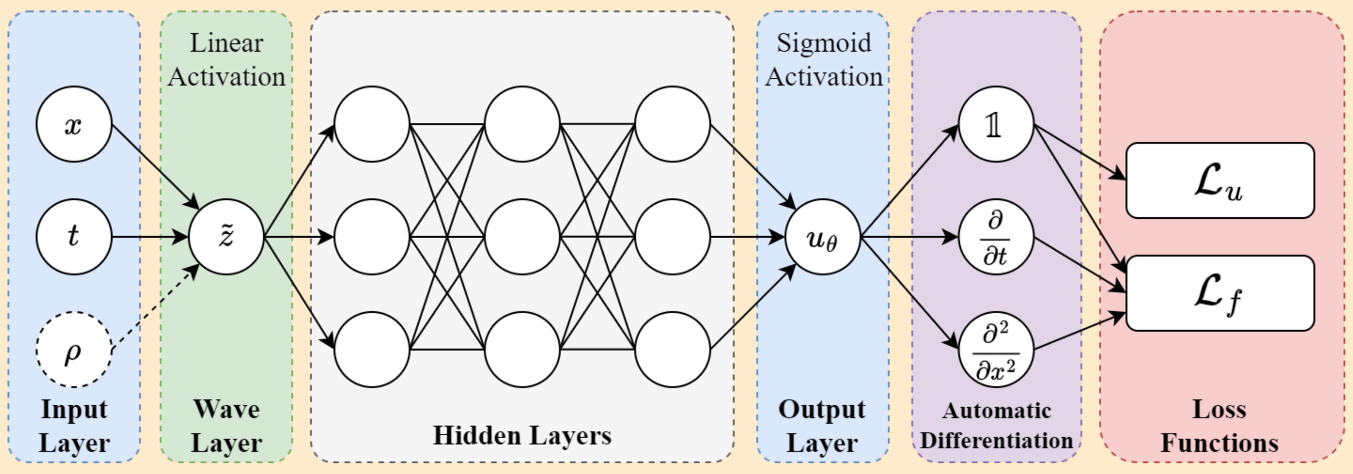
where u is the concentration of a chemical substance, and the reaction term $R(u;\rho) \equiv \rho u(1-u)$ is parameterized by the reaction rate coefficient ρ. This equation admits **traveling wave solutions** of the form $u(x,t) = u(x \pm ct) \equiv u(z)$ for every wave speed $c \geq 2\sqrt{\rho}$. For $\rho \gg 1$, the solution exhibits a sharp and fast traveling wave front which demands fine spatial and temporal resolutions when studied numerically.

Our Work

- Adapts **Physics-Informed Neural Networks (PINNs)** (Raissi et al., 2019) to solve Fisher's equation with solutions characterized by steep traveling wave fronts.
- Introduces a **residual weighting scheme** that is based on the underlying reaction dynamics and helps in tracking the propagating wave fronts.
- Explores a **specialized network architecture** tailored for solutions in the form of traveling waves.
- Evaluates the capacity of PINNs to approximate **family of solutions** to Fisher's equation.

Methodology

Physics-Informed Neural Network



- Approximates solutions to Fisher's equation in **space- and time-continuous** manner
- Minimizes loss functions during model training

$$\mathcal{L}(\theta) = \underbrace{\frac{1}{N_u}\sum_{i=1}^{N_u}\left|u^{(i)} - u_\theta^{(i)}\right|^2}_{\text{Data Loss}} + \underbrace{\frac{1}{N_f}\sum_{j=1}^{N_f}\left|\omega^{(j)} f_\theta^{(j)}\right|^2}_{\text{Physics Loss}} \tag{2}$$

- Encodes Fisher's equation through

$$f_\theta(x,t;\rho) = \frac{\partial u_\theta}{\partial t} - \frac{\partial^2 u_\theta}{\partial x^2} - \rho u_\theta(1-u_\theta) \tag{3}$$

Modifications

- ***Residual Weighting***

$$\omega = \frac{1}{\lambda|R(u_\theta;\rho)|+1} \tag{4}$$

- ***Wave Layer***

$$u_\theta(x,t;\rho) \rightarrow u_\theta(\tilde{z}) \tag{5}$$

Tested Models

Table 1: The four models employed in the study. These can be distinguished based on whether the physics loss function and/or the wave layer architecture are applied.

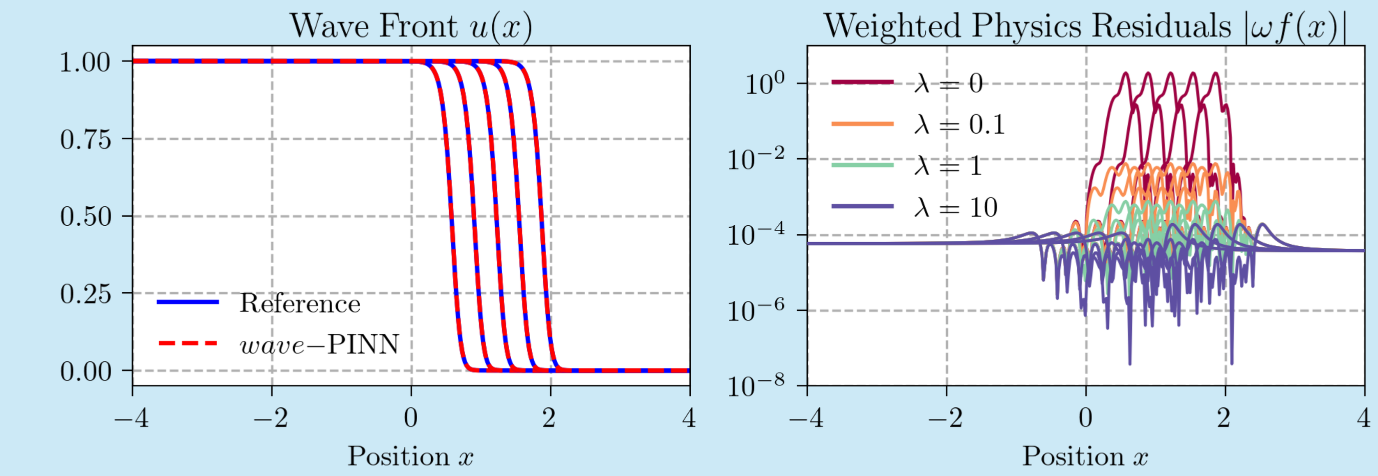
Model	Physics Loss	Wave Layer
standard-ANN		
wave-ANN		✓
standard-PINN	✓	
wave-PINN	✓	✓

Results

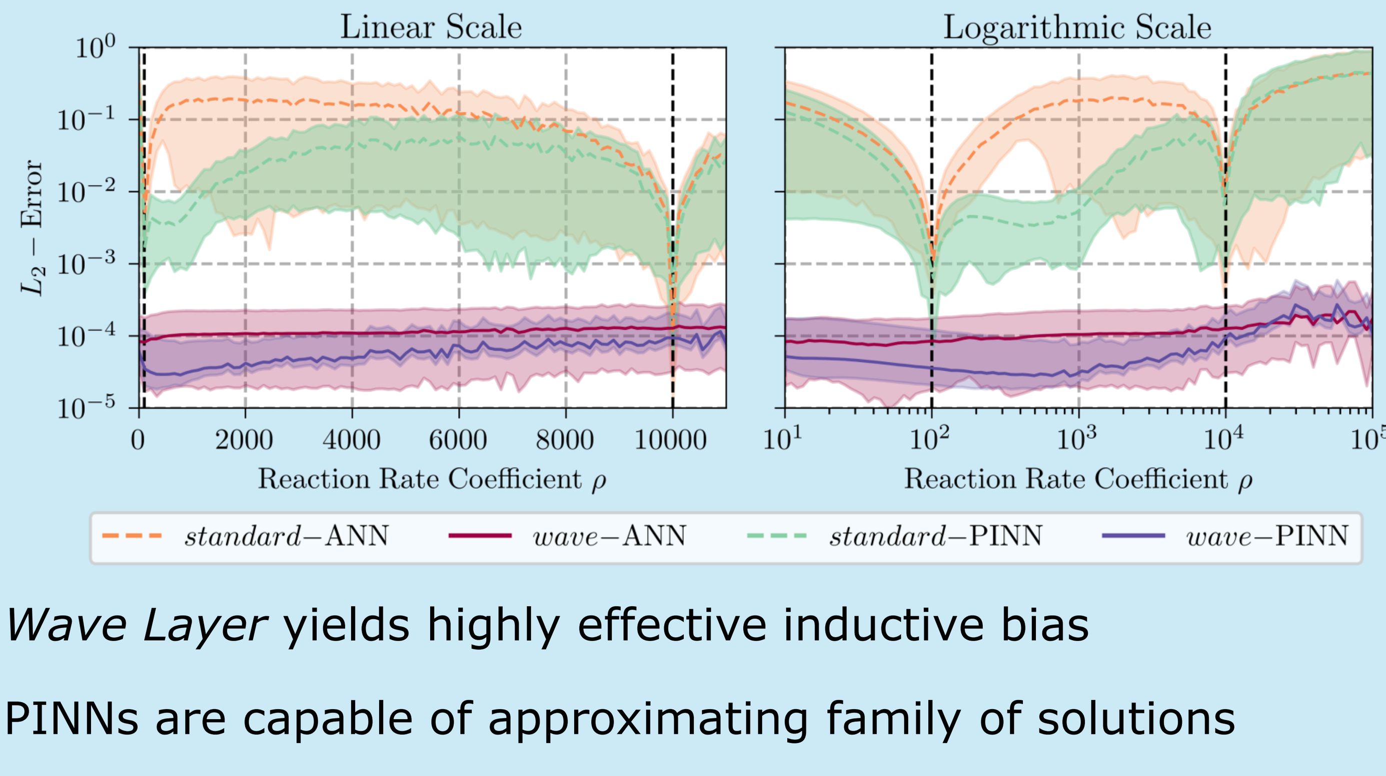
Table 2: The mean (and standard deviations) of the L_2-error for the four tested models and different values for ρ. All numbers should be multiplied by 10^{-4}. **Bold**: Best results.

Model	λ	Reaction rate coefficient, ρ: 100	1.000	10.000
standard-ANN	-	1.66 (1.30)	0.79 (0.29)	1.69 (2.10)
wave-ANN	-	1.12 (0.60)	1.05 (0.81)	1.54 (1.62)
standard-PINN	0	180 (46)	320 (184)	5689 (3469)
	0.1	37.9 (11.0)	81.1 (15.9)	36.8 (34.3)
	1	5.74 (0.95)	14.0 (3.0)	132 (66)
	10	1.72 (1.23)	8.79 (1.46)	354 (174)
wave-PINN	0	16.5 (13.5)	178 (160)	5082 (4106)
	0.1	1.26 (1.49)	1.65 (0.95)	12.1 (6.5)
	1	**0.82 (0.95)**	**0.60 (0.63)**	**1.10 (0.80)**
	10	2.16 (3.28)	0.83 (1.45)	3.11 (3.71)

- *Residual Weighting* and *Wave Layer* show outstanding improvements compared to conventional PINNs (*standard*-PINN, $\lambda = 0$)



- *Residual Weighting* helps in tracking steep wave fronts



- *Wave Layer* yields highly effective inductive bias
- PINNs are capable of approximating family of solutions

References

- Fisher, 1937, *The wave of advance of advantageous genes*, Annals of eugenics 7 (4) (1937) 355–369.
- Raissi et al., 2019, *Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations*, Journal of Computational physics 378 (2019) 686–707.