Learning to Reconstruct From Binary Measurements Alone

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TMLR Featured Paper

Problem Setup

$$y_i = \text{sign}(A_{g_i} x_i) \quad \text{for} \quad i = 1, \dots, N$$

where $y_i \in \{-1, 1\}^m$ measurement

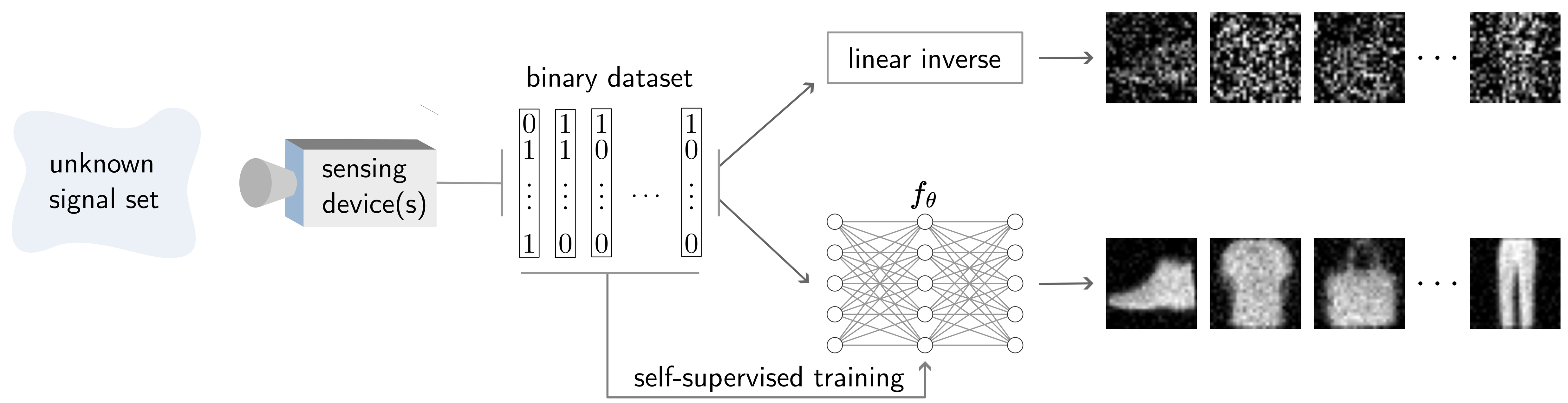
$x_i \in \mathcal{X} \subset \mathbb{S}^{n-1}$ signal

$A_g \in \mathbb{R}^{m \times n}$ forward operator

$g_i \in \{0, 1, \dots, G\}$

Invariant $\mathcal{X}$: $A_g = AT_g$ where T_g are translations, rotations, etc.

Applications: 1-bit matrix completion, 1-bit compressed sensing, etc.

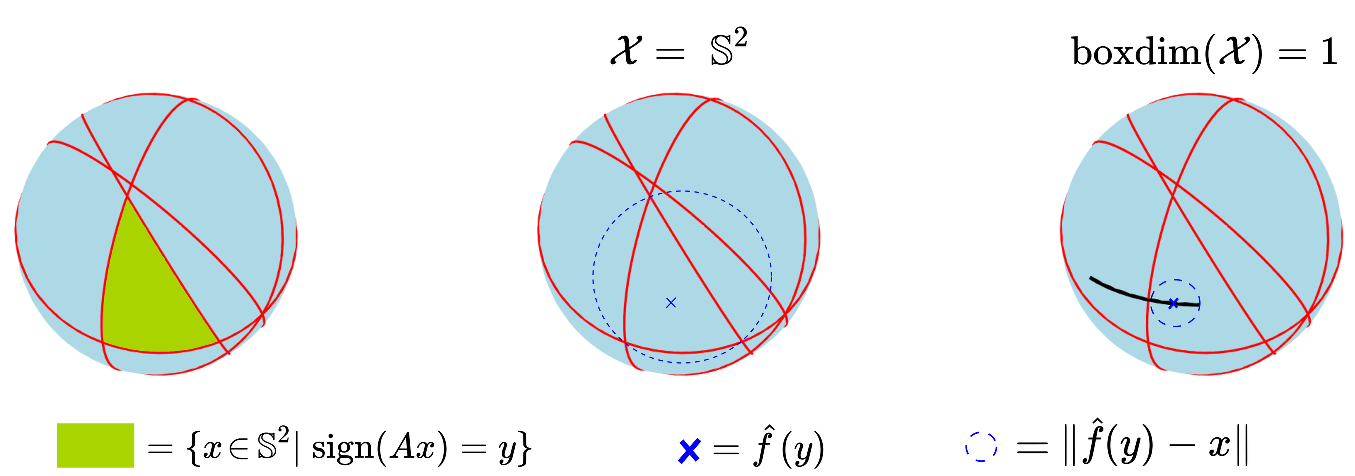


Signal Reconstruction

Can we recover a signal from its binary observation with known $\mathcal{X}$?

Optimal reconstruction:

$$\hat{f}(y) = \text{centroid}\left(\{x : \text{sign}(Ax) = y\} \cap \mathcal{X}\right)$$



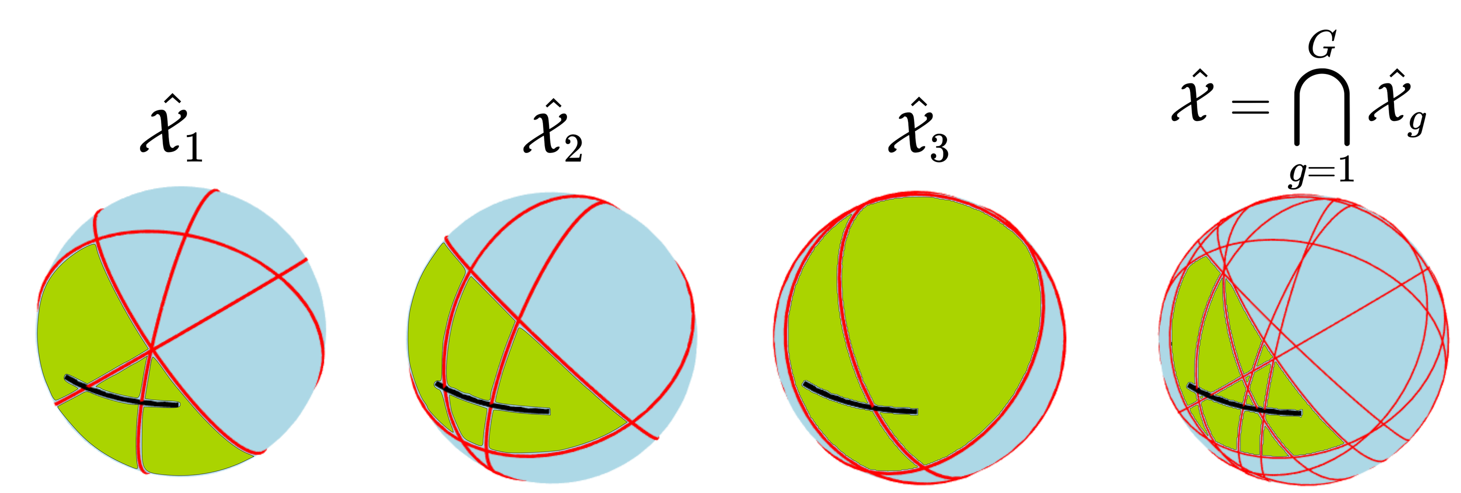
Theorem: $A \sim \mathcal{N}(0, I_{m\times n})$ *and* $\text{boxdim}(\mathcal{X}) = k$

$$\|\hat{f}(y) - x\| \lesssim \frac{k}{m} \log \frac{nm}{k}$$ *with high probability.*

Model Identification

Can we identify the signal set $\mathcal{X}$ from $\{\text{sign}(A_g \mathcal{X})\}_{g=1}^G$?

Estimation of signal set from g-th operator: $\hat{\mathcal{X}}_g = \{v \in \mathbb{S}^{n-1} : \exists x_g \in \mathcal{X},\ \text{sign}(A_g v) = \text{sign}(A_g x_g)\}$



Identification error: $\min\{\delta \geq 0 : \hat{\mathcal{X}} \subseteq \mathcal{X}_\delta\}$ where $\mathcal{X}_\delta = \{v \in \mathbb{S}^{n-1} : \inf_{x \in \mathcal{X}} \|x - v\| < \delta\}$

Theorem (lower bound): for any $A \in \mathbb{R}^{m \times n} \Rightarrow \delta > \frac{2}{3} \frac{n}{mG}$

Theorems: for $A_g \sim \mathcal{N}(0, I_{m\times n})$ *and* $\text{boxdim}(\mathcal{X}) = k$ *with high probability*

(upper bound)

$$\delta \lesssim \frac{k + n/G}{m} \log \frac{nm}{k + n/G}$$

(sample complexity)

$$N \lesssim G \left(\frac{m\sqrt{n}}{k}\right)^k$$

Self-Supervised Learning

Goal: learn reconstruction network $\hat{x} = f_\theta(y)$ from $\{y_i\}_{i=1}^N$

Multiple operators:

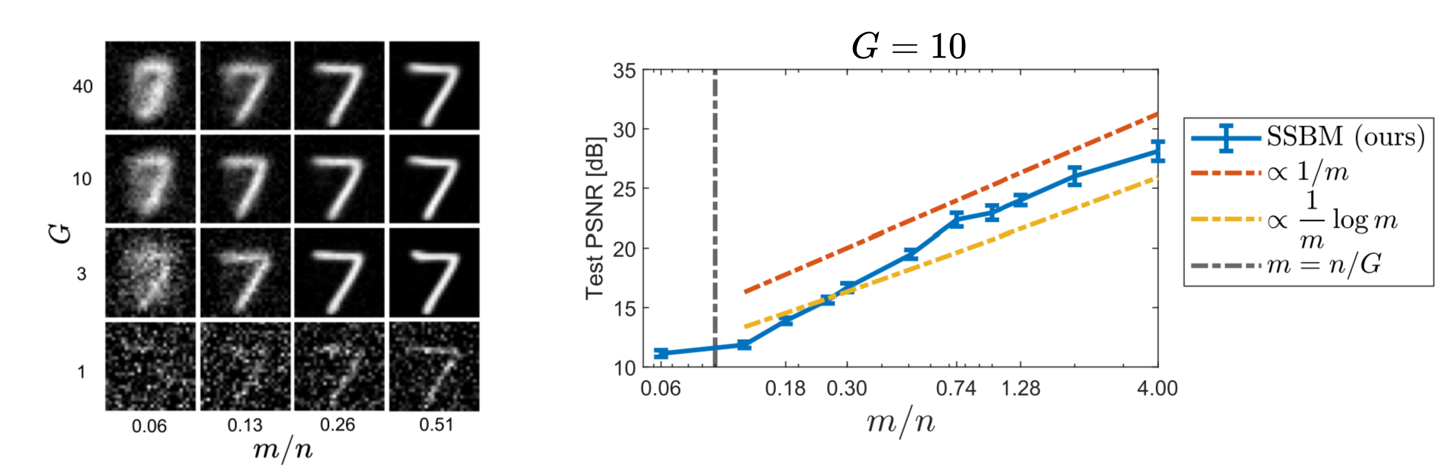
$$\mathcal{L}(\theta) = \sum_{i=1}^{N} \text{softmargin}\left(y_i, A_{g_i} f_\theta(y_i, A_{g_i})\right) + \alpha \sum_{s=1}^{G} \|\hat{x}_{i,\theta} - f_\theta(A_s \hat{x}_{i,\theta})\|^2 \quad \text{with} \quad \hat{x}_{i,\theta} = f_\theta(y_i, A_{g_i})$$

Single operator and invariant $\mathcal{X}$:

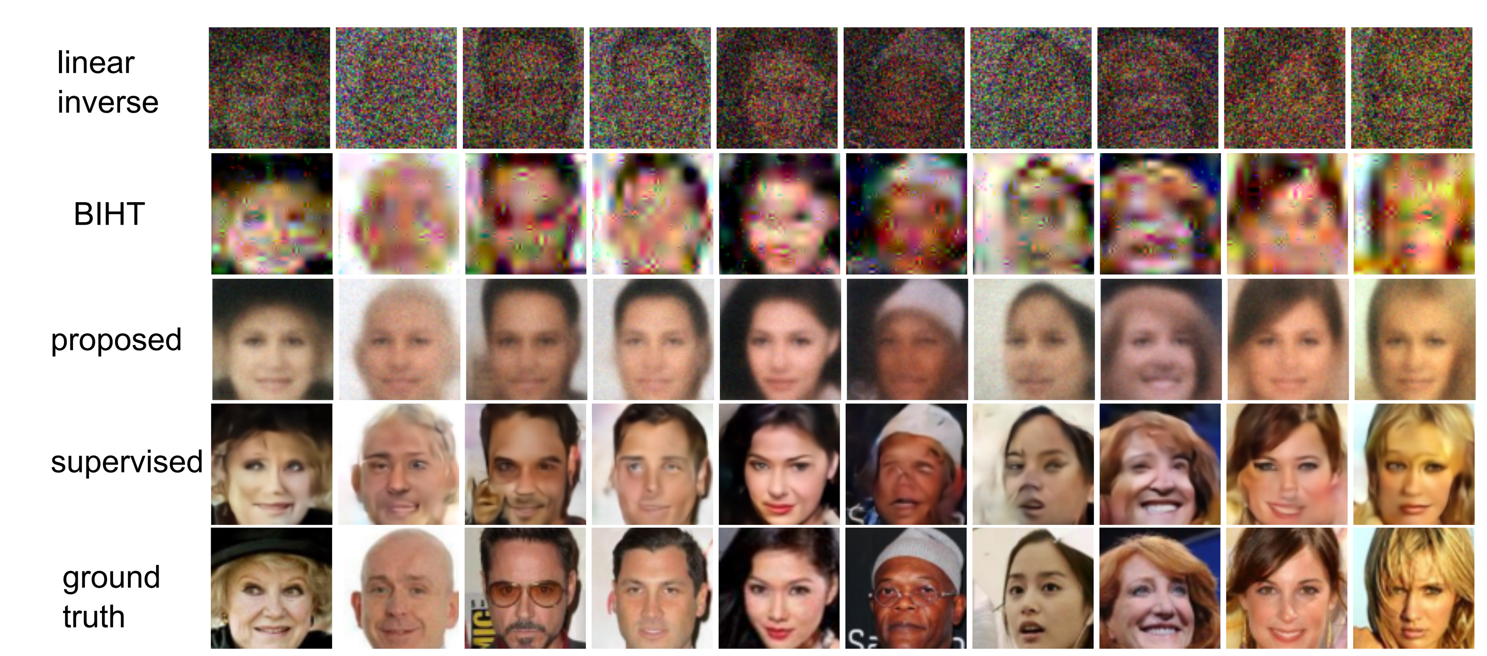
$$\mathcal{L}(\theta) = \sum_{i=1}^{N} \text{softmargin}\left(y_i, A f_\theta(y_i)\right) + \alpha \sum_{s=1}^{G} \|T_s \hat{x}_{i,\theta} - f_\theta(A T_s \hat{x}_{i,\theta})\|^2 \quad \text{with} \quad \hat{x}_{i,\theta} = f_\theta(y_i)$$

Experiments

MNIST dataset:



CelebA dataset: $G = 10,\ m = 0.2n$



References

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