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Bridging Empirics and Theory: Unveiling Asymptotic Universality Across Gaussian and Gaussian Mixture Inputs in Deep Learning

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I. Introduction

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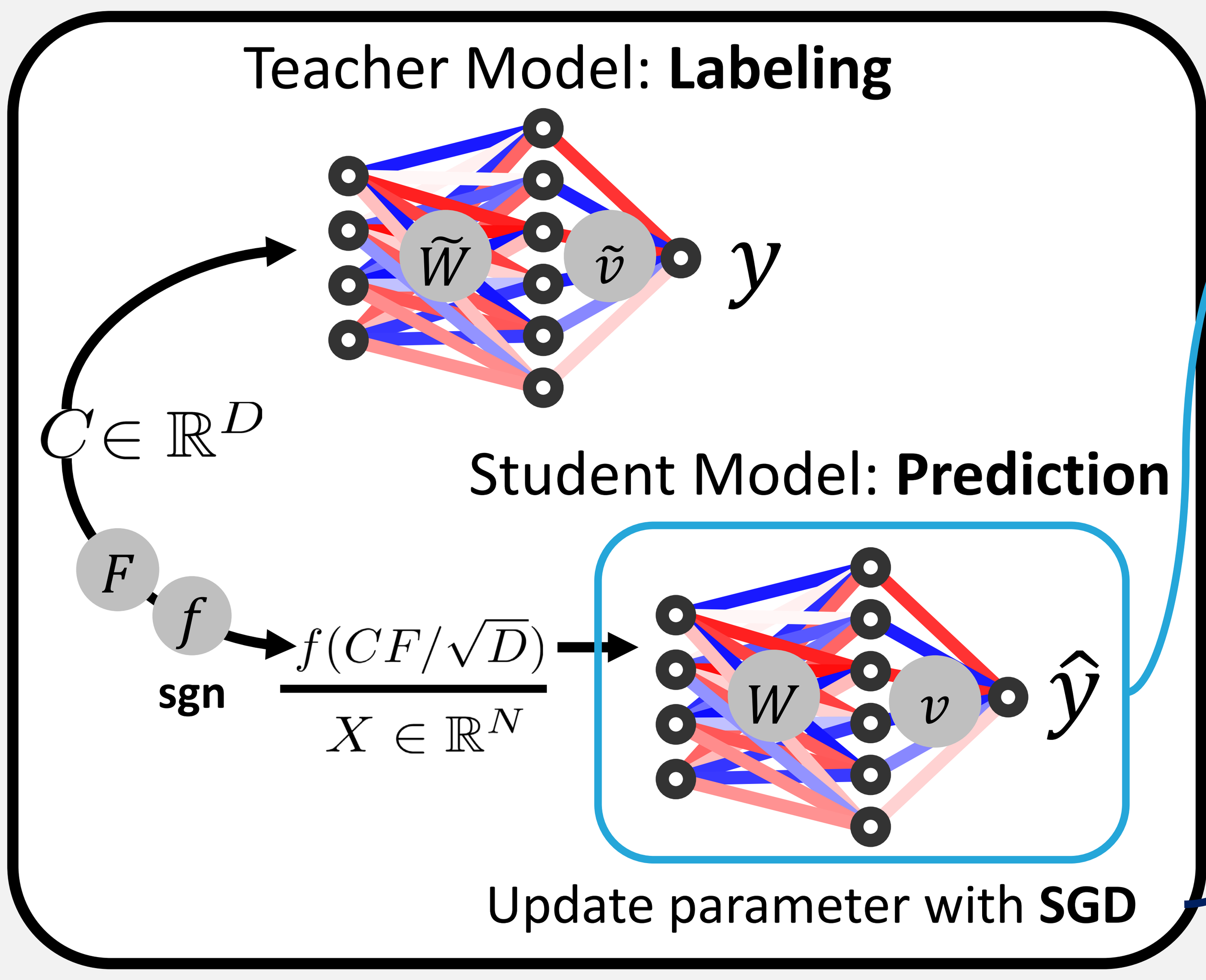
Previous Findings: Dynamics of Hidden Manifold Teacher-Student Model

The **core property C** is based on the **latent space(*manifold*)** dimension D.

In real world, this core property becomes **wrapped, resulting to X**, on the **real space** dimension N.

$C \sim \mathcal{N}(0, I)$



Gaussian Equivalence Property

In the Limit of $N \to \infty, D \to \infty$, the preactivations of the teacher and student model $\lambda = XW^\top/\sqrt{N}$, $\nu = C\widetilde{W}^\top/\sqrt{D}$ conform jointly Gaussian variables.

Statistics involving $\{\lambda, \nu\}$ are entirely represented by their mean and convariances

$Q_{k,\ell} \equiv \mathbb{E}[\lambda_k \lambda_\ell]$, $R_{k,m} \equiv \mathbb{E}[\lambda_k \nu_m]$, $\cdots$

SGD to ODE + Change of Basis

Dynamics are defined by statisitics of preactivation distribution

" The Dynamics of Student Model are fully theoretically tractable"

Ex) v_k dynamics

$$W_{k,i} := W_{k,i} - \frac{\eta}{\sqrt{N}} v_k (\hat{y} - y) g'(\lambda_k) f(U_i)$$

$$v_k := v_k - \frac{\eta}{N} g(\lambda_k)(\hat{y} - y)$$

Scaled SGD

$\Delta t \to dt$

$$\frac{dv_k}{dt} = \eta \left[\sum_n^M \tilde{v}_n I_2(k,m) - \sum_j^K v_j I_2(k,j) \right]$$

Expectation of functions

$$I_2(k,m) = \mathbb{E}[g(\lambda_k)\tilde{g}(\nu_n)]$$

$$I_2(k,j) = \mathbb{E}[g(\lambda_k)g(\lambda_j)]$$

Tractable under GEP;

$\because$ ***$\{\lambda, \nu\}$ Statistics are tractable***

Q : The Simple Gaussian $C \sim \mathcal{N}(0, I)$ setting is enough?

How dynamics changes as the inherent distribution deviate from simple Gaussian to Gaussian mixture?

II. Setting

Gaussian Mixture Setting

- m component Gaussian mixture with moderating parameter α

$$r = r_i \sim \mathcal{N}(\mu_i, I), \quad \mu_i \sim \mathcal{U}[-\alpha, \alpha), \text{ with } p_i, \sum^m p_i = 1$$

Empirical investigations across a spectrum (m, α) of Gaussian mixture

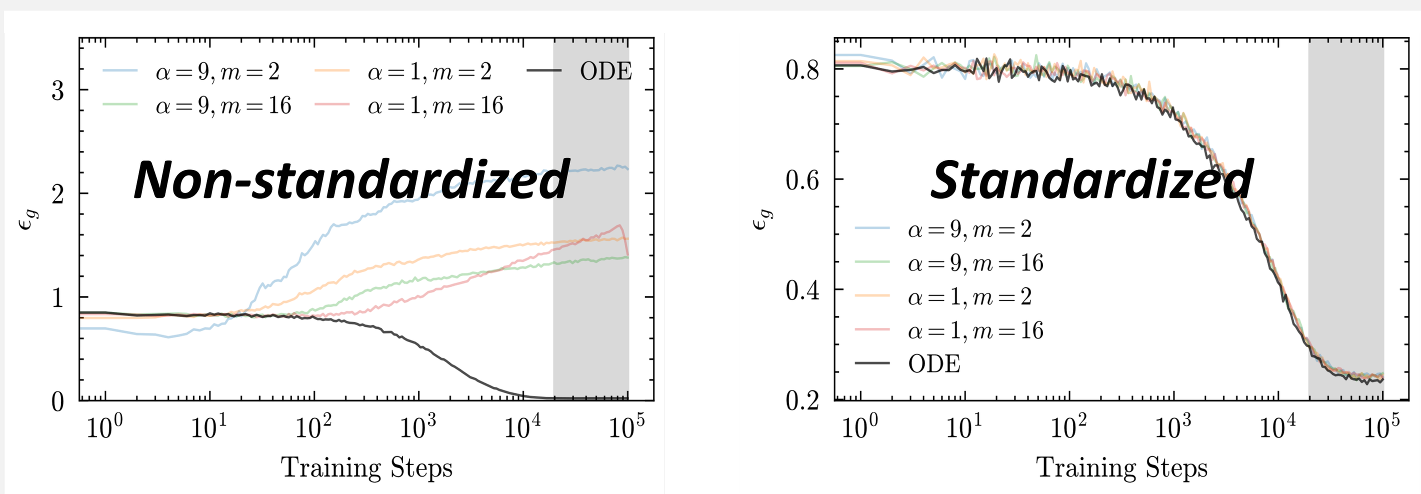
Dynamic Metrics

- Covariance of preactivations $Q_{k,\ell}, R_{k,m}$,
- generalization error ϵ_g and 2nd layer weight v_k

III. Results & Discussion

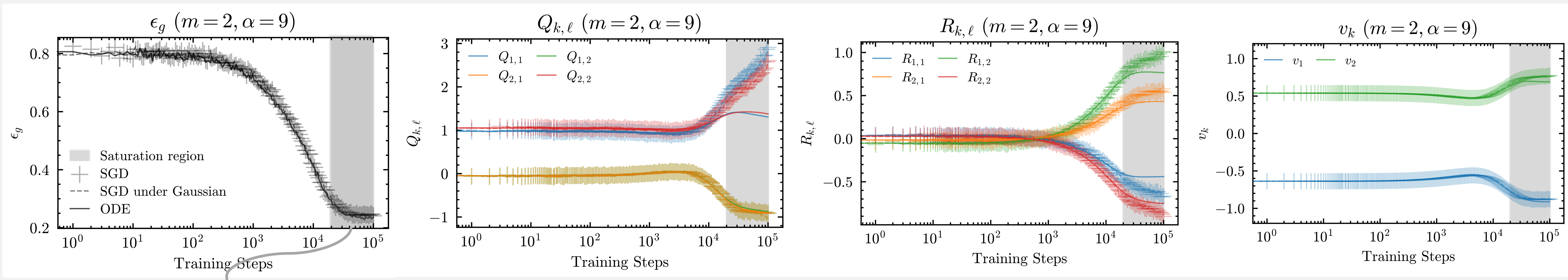
Dynamics across various Gaussian mixture

- With ***non-standardized*** Gaussian mixture, the theoretical prediction completely **collapse** for ***all spectrum of mixture***
- With ***standardized*** Gaussian mixture***, mixture spectrum dose not effect***



Dynamics under standardized Gaussian mixture

Sample-wise $C \sim \mathcal{P}$, $\quad C := (C - \mathbb{E}[C])/\sqrt{\mathbb{E}[(C - \mathbb{E}[C])^2]}$.



Saturation region; Randomness has a significant effect

- Theoretical prediction under the Simple Gaussian **surprisingly aligned** with Mixture dynamics

Convergence in Standardized Gaussian Mixture

Key quantity: "expectation value of functions" in dynamics

Dominance of Moments in the Expectation value of Functions

[Definition] If $\mathcal{P}$ shares identical cumulants with $\mathcal{D}$ up to order 2, represent $\mathcal{P}$ as $\mathcal{P}_{\mathcal{D}_2}$

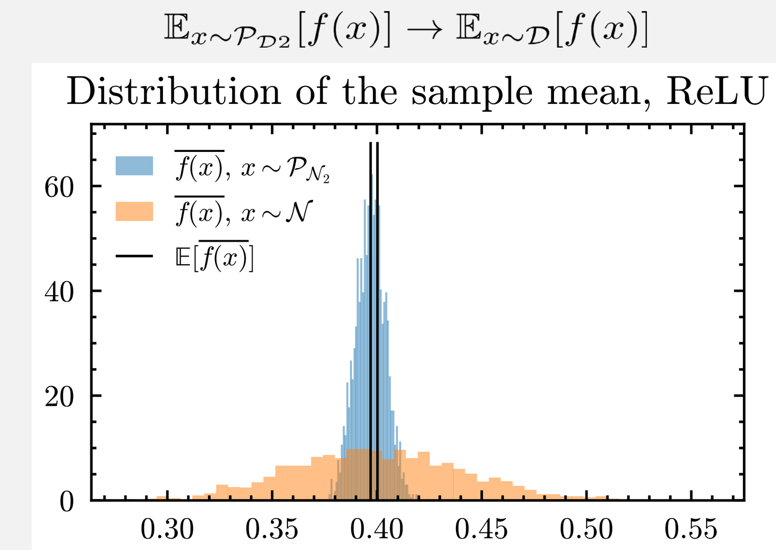
[Lemma] If a function f has the property of **erasing the influence of high-order cumulants**, then the expectation value of the function tends towards the same value.

$$\mathbb{E}_{x \sim \mathcal{P}_{\mathcal{D}2}}[f(x)] \to \mathbb{E}_{x \sim \mathcal{D}}[f(x)]$$

[Proof] Taylor expansion of expectation + Function property vanish high order derivation

$$\mathbb{E}_{x \sim \mathcal{P}_{\mathcal{D}_2}}[f(x)]_{x \in I} = \mathbb{E}[f(\mu) + \cdots + f^{(n)}(\mu)\frac{(x-\mu)^n}{n!} + \cdots]_{x \in I} \approx f(\mu) + f''(\mu)\mathbb{E}[\frac{(x-\mu)^2}{2!}] \to \mathbb{E}_{x \sim \mathcal{D}}[f(x)]_{x \in I}$$

$\mathbb{E}_{x \sim \mathcal{P}_{\mathcal{D}2}}[f(x)] \to \mathbb{E}_{x \sim \mathcal{D}}[f(x)]$



Emprical results, with $f \equiv$ ReLU

[Lemma] ReLU activation function **well erasing** influence of high-order cumulants

[Proof] In piecewise view, ReLU function is zero in 2nd Derivative

IV. Summary

Specific function + standardized ⇒ Make Dynamics dominantly up to 2nd Cumulants.

Even the mixture, the dynamics also tend to approximate same.

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