Schur's Positive-Definite Network: Deep Learning in the SPD cone with structure

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Can Pouliquen

Mathurin Massias

Titouan Vayer

Ecole Normale Supérieure de Lyon
Inria Lyon
Lyon, FRANCE

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Challenge: design SPD-to-SPD neural architectures

SPD matrices everywhere →
- Graph Learning (Shrivastava et al., 2019)
- Computer Vision (Gao et al., 2019)
- Optimal Transport (Flamary et al., 2019)

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$\Longrightarrow$ can neural nets learn them from data ?

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Difficult !

Idea: Leverage inductive bias

Unroll iterative algorithms !

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STILL CHALLENGING:

- How to handle additional constraints?
- No reliable way to ensure both SPD and sparse outputs (hard problem (Guillot and Rajaratnam, 2015))

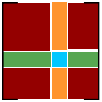
Contribution: Schur's Positive-Definite Network

Setup: Block matrices $\mathbf{\Theta} = \begin{bmatrix} \mathbf{\Theta}_{11} & \boldsymbol{\theta}_{12} \\ \boldsymbol{\theta}_{12}^\top & \theta_{22} \end{bmatrix}$

$\mathbf{\Theta}_{11}$

$\boldsymbol{\theta}_{12} = \boldsymbol{\theta}_{21}^\top$

θ_{22}

$\mathbf{\Theta} =$ 

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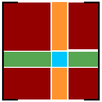
$\boldsymbol{\theta}_{12} = \boldsymbol{\theta}_{21}^\top$ $\quad \boldsymbol{\Theta} =$ 

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Key ingredient: Schur's condition for positive-definiteness

$$\boldsymbol{\Theta}_{\mathbf{11}} \succ 0$$

$$\theta_{22} - \boldsymbol{\theta}_{\mathbf{12}}^\top [\boldsymbol{\Theta}_{11}]^{-1} \boldsymbol{\theta}_{12} > 0$$

Key observation: Holds for any column value !

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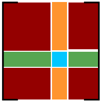
$\theta_{12} = \theta_{21}^\top$ $\Theta =$ 

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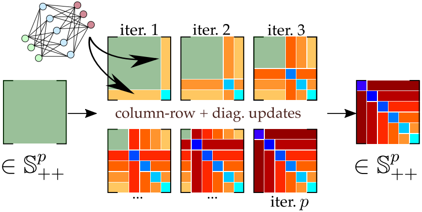
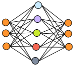
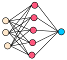
Core idea: flow through column/row/diagonal updates



$\theta_{12}^+ = \theta_{21}^{\top +} =$ 

$\theta_{22}^+ =$  $+ \theta_{21}^+ [\Theta_{11}]^{-1} \theta_{12}^+$

For more details...

Hope to see you at the poster session ! :-)

Check out the paper:

H. Shrivastava, X. Chen, B. Chen, G. Lan, S. Aluru, H. Liu, and L. Song. Glad: Learning sparse graph recovery. In *ICLR*, 2019.

Z. Gao, Y. Wu, X. Bu, T. Yu, J. Yuan, and Y. Jia. Learning a robust representation via a deep network on symmetric positive definite manifolds. *Pattern Recognition*, 2019.

R. Flamary, K. Lounici, and A. Ferrari. Concentration bounds for linear monge mapping estimation and optimal transport domain adaptation. *arXiv preprint arXiv:1905.10155*, 2019.

D. Guillot and B. Rajaratnam. Functions preserving positive definiteness for sparse matrices. *Transactions of the American Mathematical Society*, 2015.