Physics-Informed Deep Inverse Operator Networks for solving PDE Inverse Problems

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Motivation

- Inverse problems involving PDEs can be seen as discovering a mapping from measurement data to unknown quantities
- Such problems are often framed within operator learning models $e.g.$, Neural Inverse Operators[1]
- However, existing methods typically rely on large amounts of labeled training data, which is impractical for most real-world applications
- Moreover, these supervised models may fail to capture the underlying physical principles accurately

[1] R. Molinaro et al., Neural Inverse Operators for Solving PDE Inverse Problems. ICML 2023

Goal

- We propose Physics-Informed Deep Inverse Operator Networks (PI-DIONs)
- Learn the solution operator for PDE-based inverse problems without labeled training data
- Ensure the trained solution operator adheres to underlying physical principles
- Establish stability estimates for the proposed operator learning framework

Inverse problems

- Consider a generic PDE

$$\mathcal{N}(u, s) = 0, \qquad x \in \Omega$$
$$\mathcal{B}(u) = 0, \qquad x \in \partial\Omega$$

- In the supervised training paradigm, one directly parametrizes the inverse operator:

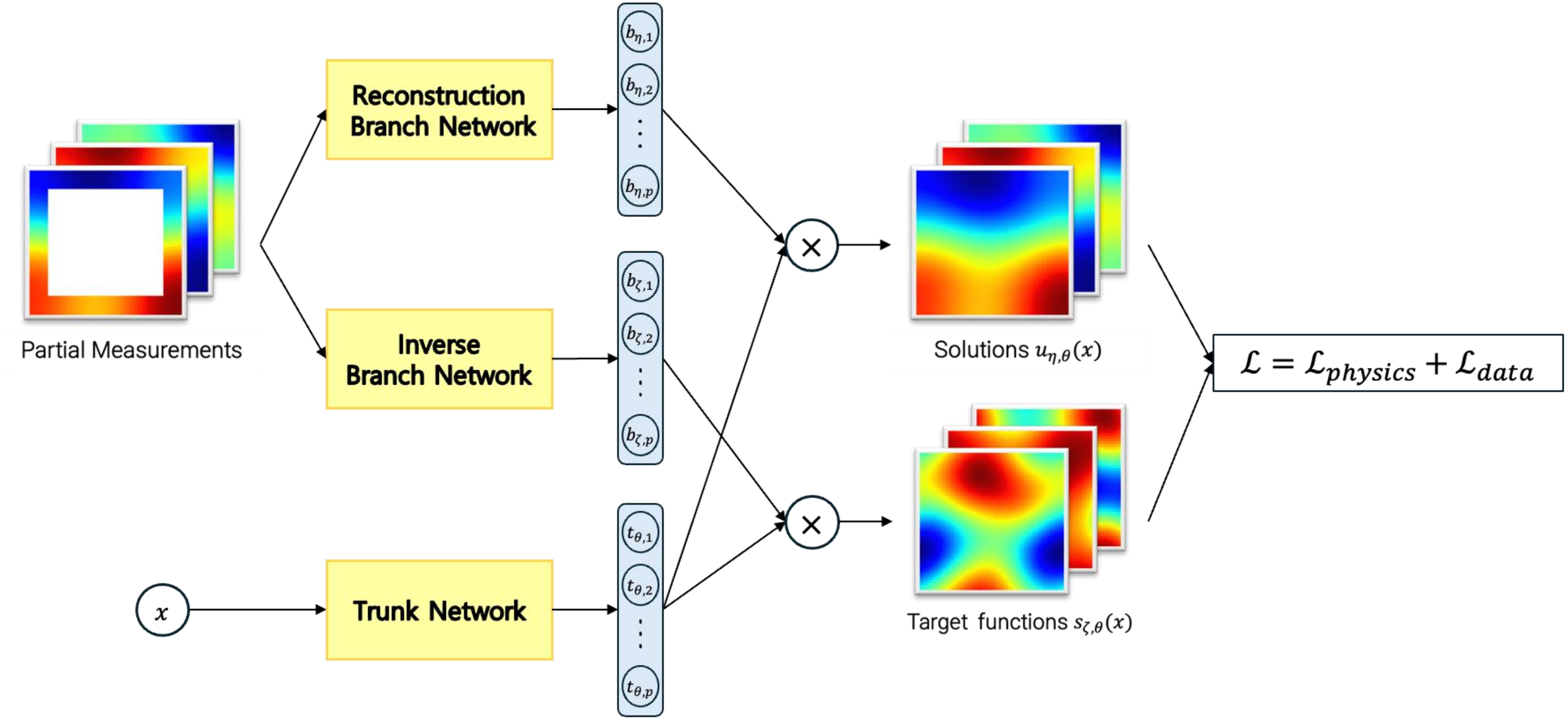
$$\mathcal{F}_\theta : u\big|_{\partial\Omega_m} \to s$$

it is hard to impose PDE loss as we do not parameterize u, s as functions of (x, t)

- We decided to employ DeepONet architecture to model u and s as functions of (x, t)

PI-DIONs

- Physics-Informed Deep Inverse Operator Networks



Experiments

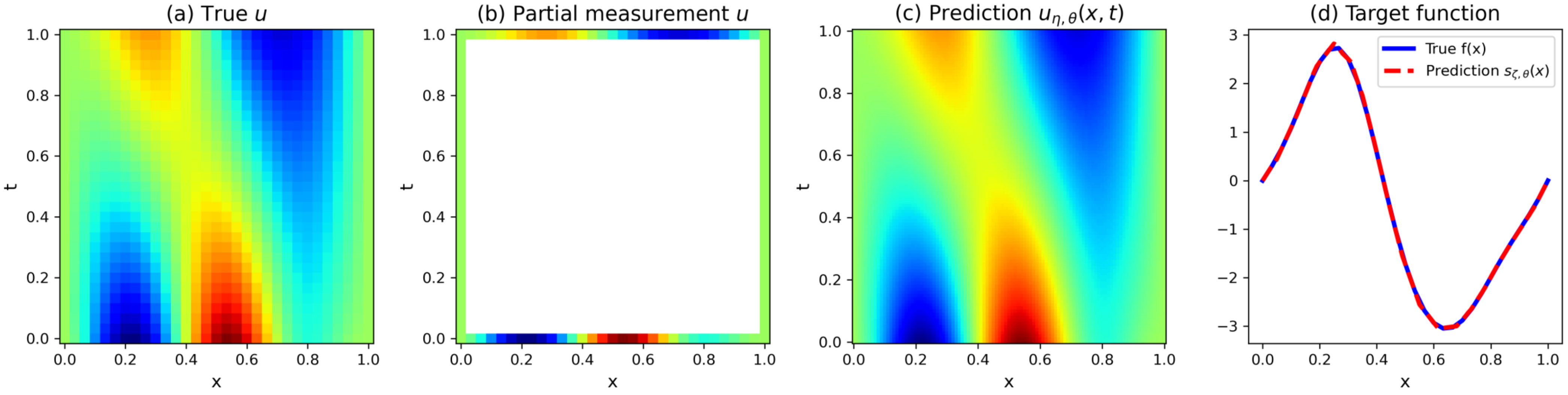
- Inverse source problem using boundary measurement for the reaction-diffusion equation

$$\begin{cases} u_t + \mathcal{L}u = f(x) \quad (x,t) \in \Omega \times (0,T] \\ u(x,t) = b(x,t), (x,t) \in \partial\Omega \times (0,T] \\ u(x,0) = u_0(x),\ x \in \Omega \end{cases}$$

- In many engineering configurations, the internal source $f(x)$ is very hard to measure
- The inverse source problem amounts to approximate $f(x)$ using $u(x,0), u(x,T), b(x,t)$, and the reaction diffusion equation

Experiments

- We trained the PI-DIONs using 1,000 training samples of $u|_{\partial\Omega_T}$ to predict $f(x)$ and tested on unseen 1,000 samples
- PI-DIONs learns both $\mathcal{I} : u|_{\partial\Omega_T} \to u$ and $\mathcal{J}: u|_{\partial\Omega_T} \to f$
- The trained model can predict the solution and the source function in arbitrary resolution



Experiments

- Relative test errors

Model	# Train data	Reaction Diffusion	Darcy Flow	Helmholtz equation
DeepONet	50	33.60%	20.15%	64.17%
w/ MLP, CNN branch	500	1.29%	9.06%	23.56%
(Supervised)	1000	1.10%	8.77%	7.86%
DeepONet	50	42.78%	28.79%	74.45%
w/ FNO branch	500	32.23%	12.78%	23.61%
(Supervised)	1000	21.10%	9.12%	11.30%
FNO (Supervised)	50	N/A	10.11%	38.42%
	500	N/A	5.59%	28.23%
	1000	N/A	4.41%	25.85%
PI-DION(Ours) (Supervised)	1000	1.04%	**3.45%**	**5.64%**
PI-DION(Ours) (Unsupervised)	1000	**1.03%**	8.10%	8.05%

Theoretical results

- We showed that the stability estimate for the PINNs can be extended to PI-DIONs where $u_{\eta,\theta}$ and $s_{\zeta,\theta}$ denote the output of PI-DIONs

Theorem 3. *Suppose the same sampling process holds as in Theorem 2. Additionally, assume that the number of sampled functions N and the number of grid points L, K, M satisfy the conditions in Theorem 2. Then, for any $u \in \mathcal{U}$, $s \in \mathcal{S}$, and $\varepsilon > 0$, the following inequality holds with probability at least* $(1-2\delta)(1-2\sqrt{\epsilon}-\frac{\mathcal{L}_{physics}+\mathcal{L}_{data})}{\sqrt{\epsilon}})$,

$$\|u_{\eta,\theta}-u\|_{L^2(\Omega)}+\|s_{\zeta,\theta}-s\|_{L^2(\Omega)} \leq \sqrt{\varepsilon}.$$

Conclusion

- We proposed a novel architecture, called PI-DIONs, for learning the solution operator of the PDE inverse problems
- Numerical studies demonstrate that the proposed method achieved test errors comparable to those of traditional supervised baselines, thereby indicating that unsupervised training can be as effective as its supervised counterpart in certain contexts
- Theoretical results show that PI-DIONs can accurately predict both the solution u and the unknown quantity s based on partial measurement data $u|_{\Omega_m}$