Approximating **Tame functions** (NNs) by piecewise polynomials: **Theoretical bound** & **Numerical results**

Piecewise Polynomial Regression of Tame Functions via Integer Programming

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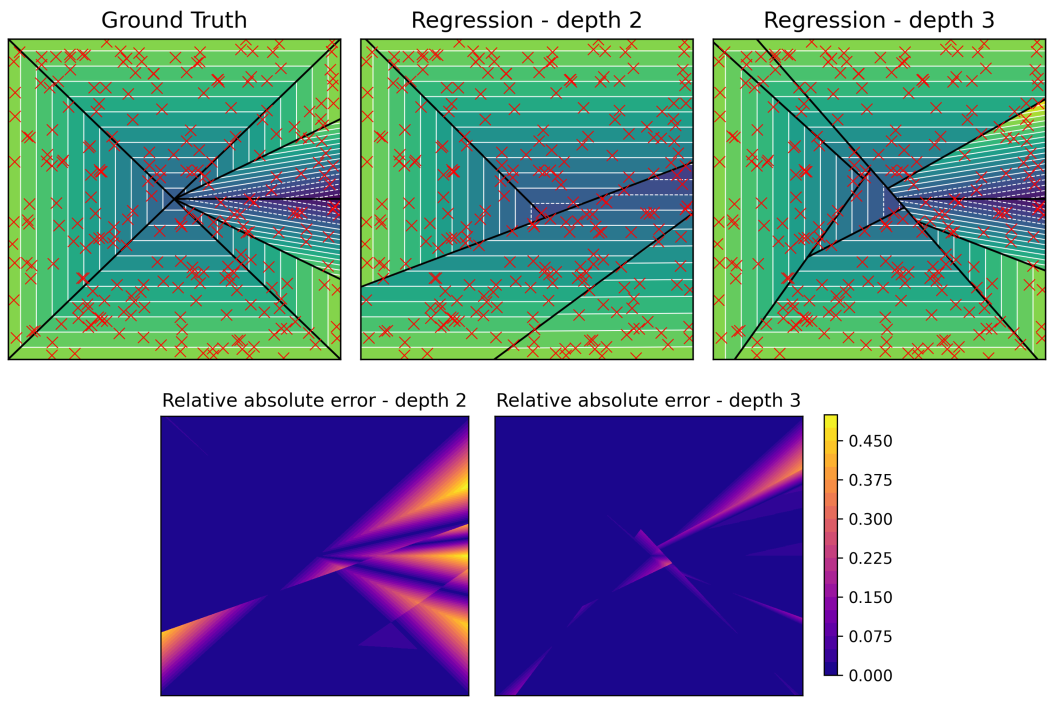
Tame Functions

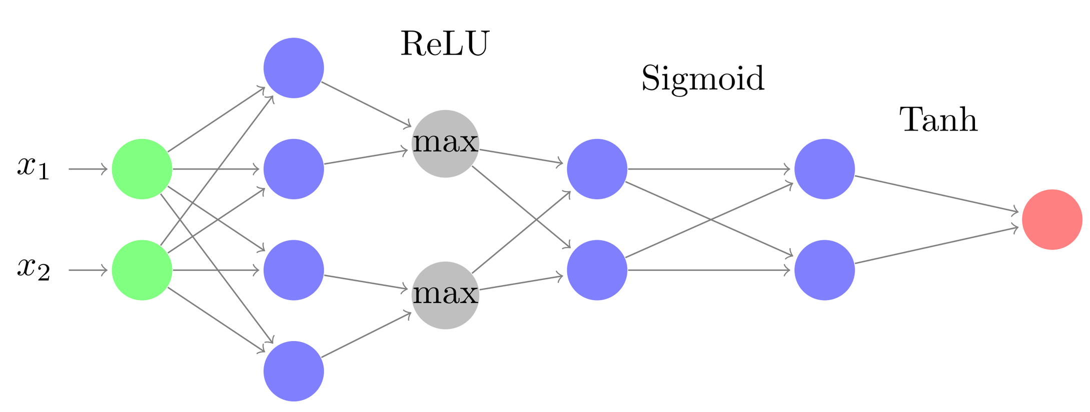
Tame functions are a general class of nonsmooth and nonconvex functions. They appear in a broad range of useful and difficult applications, such as:

- common deep learning architectures [8, 6],
- empirical risk minimization frameworks [9],
- in mixed-integer optimization, with the value function and the solution to the so-called subadditive dual [2],
- in quantum information theory, with approximations of the matrix exponential for a k-local Hamiltonian [7, 1],
- and in quantum chemistry, with functions describing the electronic structure of molecules.

An example tame function ("cone") with seven strata (left) and its approximation and error



A Neural Network example of a tame function:



Approximation bound

When approximating a m-times continuously *differentiable function* f, the best degree N polynomial incurs an $\mathcal{O}(1/N^m)$ error [3].

For *nonsmooth functions*, the rate is notably worse. For example, approximating the absolute value over the interval $[-1, 1]$ incurs *exactly* the slow rate $1/N$ [4]:

$$\inf_{p \in \mathcal{P}_N} \| |\cdot| - p \|_{\infty,[-1,1]} = \frac{\beta}{N} + o\left(\frac{1}{N}\right).$$

Thus, to approximate **Tame functions**, polynomials are not efficient. Denote $\widetilde{\mathcal{P}}_N^l(A)$ a set of *piecewise* polynomials on A, such that

1. each piece is a polyhedron defined as the intersection of l halfspaces, represented as a leaf of a complete binary tree of depth l,
2. the restriction of the function to each piece is a polynomial of degree at most N.

Theorem 1. *Consider a function $f : A \to \mathbb{R}$, a qualified* (*Asm. 1 in the paper*) *$\mathcal{C}^m$-stratification of f for some $m \geq 2$ such that:*

- *f is a **tame** function,*
- *A is a connected compact subset of $[0, 1]^n$,*
- *f is continuous over A.*

*Then f can be approximated by **piecewise polynomial functions**:*

$$\inf_{p \in \widetilde{\mathcal{P}}_N^l(A)} \|f - p\|_{\infty,A} \leq C_1 N^{-m} + C_2 l^{-\frac{2}{n-1}}.$$

where $C_1 = c_1(n, m, A, f)$ and $C_2 = c_2(n, m, f)$.

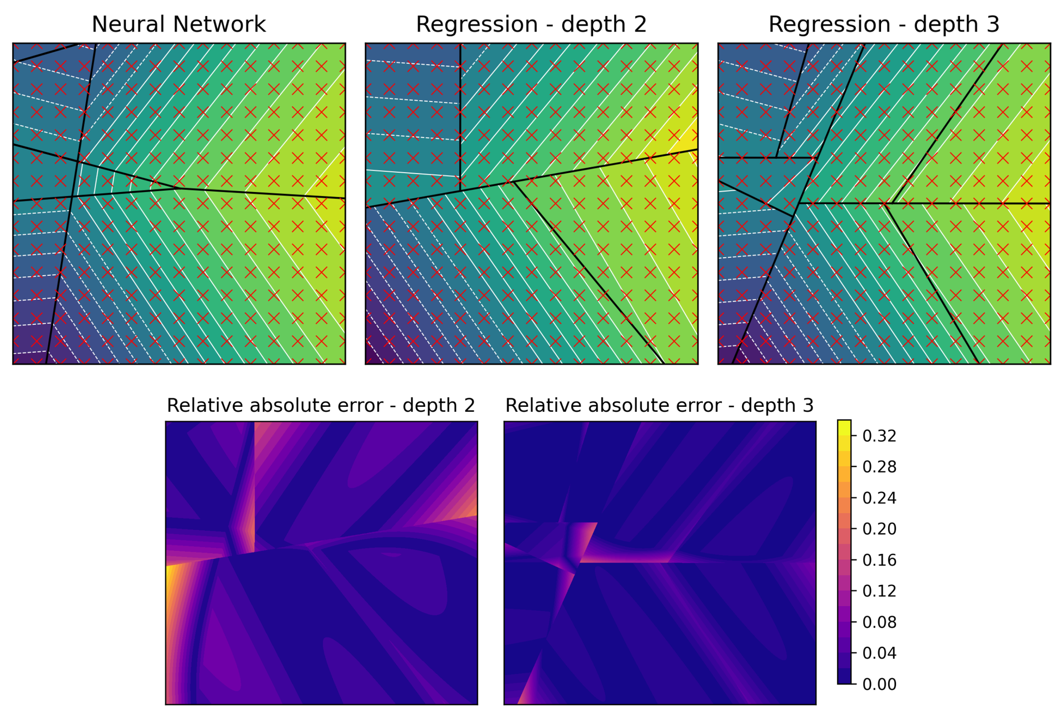
Stratification

We say that f admits a definable $\mathcal{C}^m$-stratification if there exists a finite partition of $\mathbb{R}^n$ such that

- each stratum is a definable connected $\mathcal{C}^m$-manifold,
- f is $\mathcal{C}^m$-smooth on each stratum,
- any two strata $\mathcal{M}$ and $\mathcal{M}'$ satisfy the frontier condition: $\mathcal{M} \cap \operatorname{cl} \mathcal{M}' \neq \emptyset \Rightarrow \mathcal{M} \subset \operatorname{cl} \mathcal{M}'$.

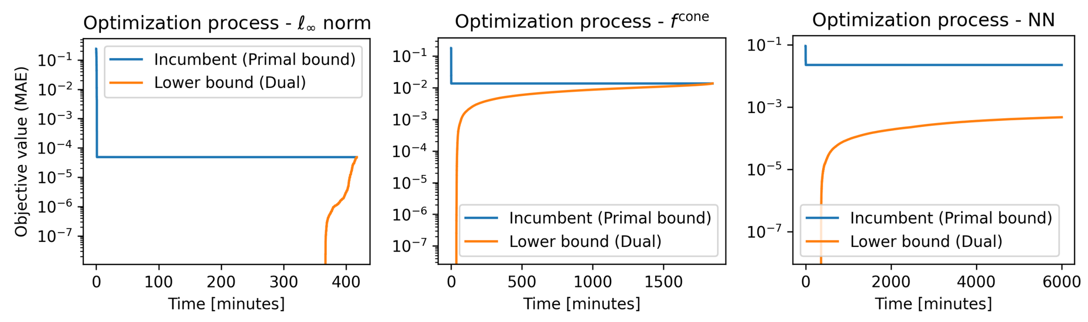
Mixed-Integer Formulation

We slightly modify the affine-hyperplane tree Mixed-Integer Linear formulation (OCT-H, [5]) to separate the input space into 2^l polyhedral regions and fit a polynomial to each region.



Solving process

The primal objective value converges quickly; most of the computation time is spent on improving lower bound.



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