Equivariant Splitting: Self-supervised learning from incomplete data

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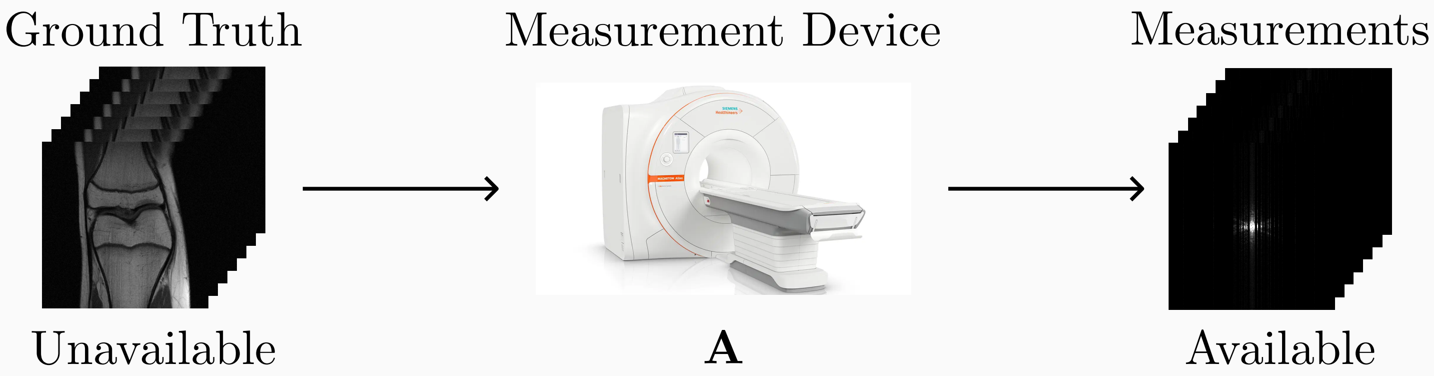
Self-supervised inverse problems

Problem setup Let $\boldsymbol{x} \in \mathcal{X} \subsetneq \mathbb{R}^n$, $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $\boldsymbol{\varepsilon} \sim \mathcal{N}(0, \sigma^2 \boldsymbol{I}_m)$

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{\varepsilon}$$

Goal find $f(\boldsymbol{y}, \boldsymbol{A}) \approx \boldsymbol{x}$

Challenge use only $(\boldsymbol{y}_i)_{i \in I}$

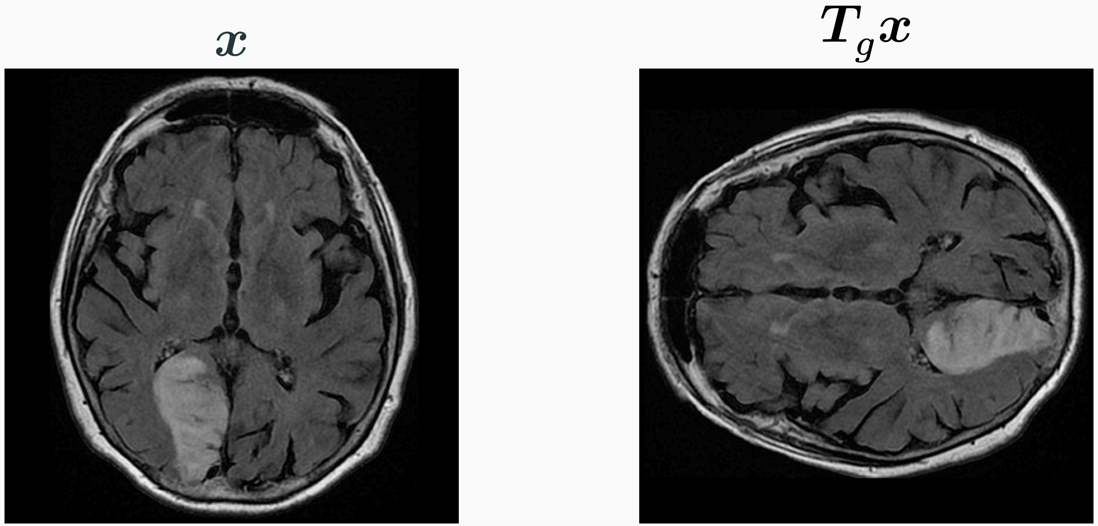


Assumption invariance of $p(\boldsymbol{x})$ to transformations

$$p(\boldsymbol{T}_g\boldsymbol{x}) = p(\boldsymbol{x}) \quad \forall \boldsymbol{x}, g \tag{1}$$

Virtual operators

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}\boldsymbol{T}_g\underbrace{\boldsymbol{T}_g^{-1}\boldsymbol{x}}_{\boldsymbol{x}'\in\mathcal{X}} = \boldsymbol{A}_g\boldsymbol{x}' \tag{2}$$

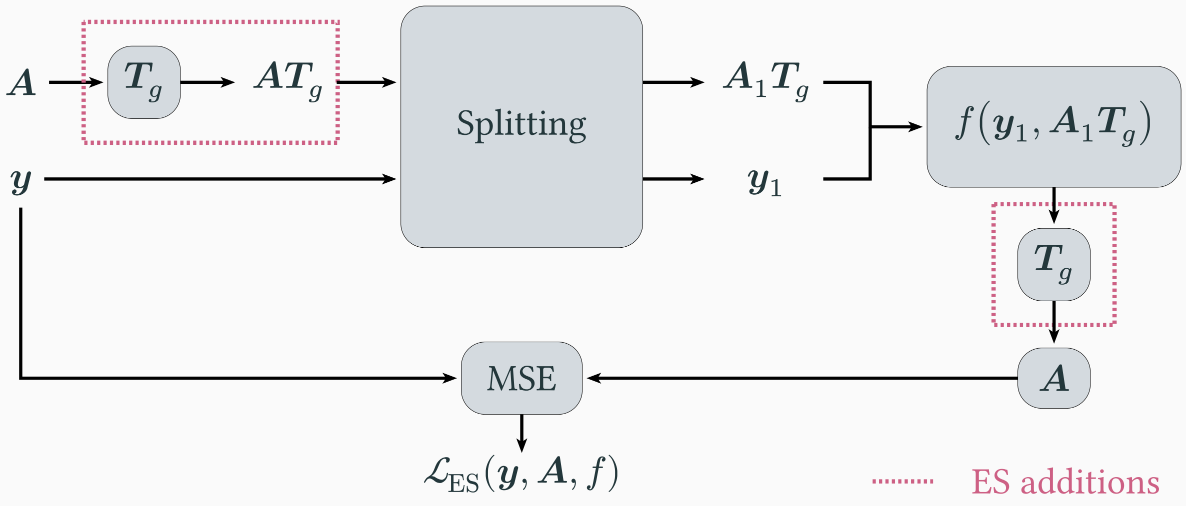


Measurement splitting

$$\boldsymbol{y} = \left(\boldsymbol{y}_1^\top, \boldsymbol{y}_2^\top\right)^\top = \left(\boldsymbol{A}_1^\top, \boldsymbol{A}_2^\top\right)^\top \boldsymbol{x} \tag{3}$$

Equivariant splitting loss (ES)

$$\mathcal{L}_{\mathrm{ES}}(\boldsymbol{y}, \boldsymbol{A}, f) \triangleq \mathbb{E}_g\left\{\mathbb{E}_{\boldsymbol{y}_1, \boldsymbol{A}_1 \mid \boldsymbol{y}, \boldsymbol{A}\boldsymbol{T}_g}\left\{\left\|\boldsymbol{A}\boldsymbol{T}_g f\left(\boldsymbol{y}_1, \boldsymbol{A}_1\boldsymbol{T}_g\right) - \boldsymbol{y}\right\|^2\right\}\right\} \tag{4}$$



Efficient loss evaluation

Theorem (Efficient loss evaluation)

If $f(\boldsymbol{y}, \boldsymbol{A})$ is an equivariant reconstructor

$$f\left(\boldsymbol{y}, \boldsymbol{A}\boldsymbol{T}_g\right) = \boldsymbol{T}_g^{-1} f(\boldsymbol{y}, \boldsymbol{A}), \quad \forall \boldsymbol{y}, \boldsymbol{A}, \boldsymbol{g} \tag{5}$$

then the ES loss reduces to

$$\mathcal{L}_{\text{ES}}(\boldsymbol{y}, \boldsymbol{A}, f) = \mathbb{E}_{\boldsymbol{y}_1, \boldsymbol{A}_1 \mid \boldsymbol{y}, \boldsymbol{A}} \left\{ \|\boldsymbol{A} f(\boldsymbol{y}_1, \boldsymbol{A}_1) - \boldsymbol{y}\|^2 \right\} \tag{6}$$

Examples of equivariant reconstructors

- **Reynolds averaging** for any reconstructor $r(\boldsymbol{y}, \boldsymbol{A})$

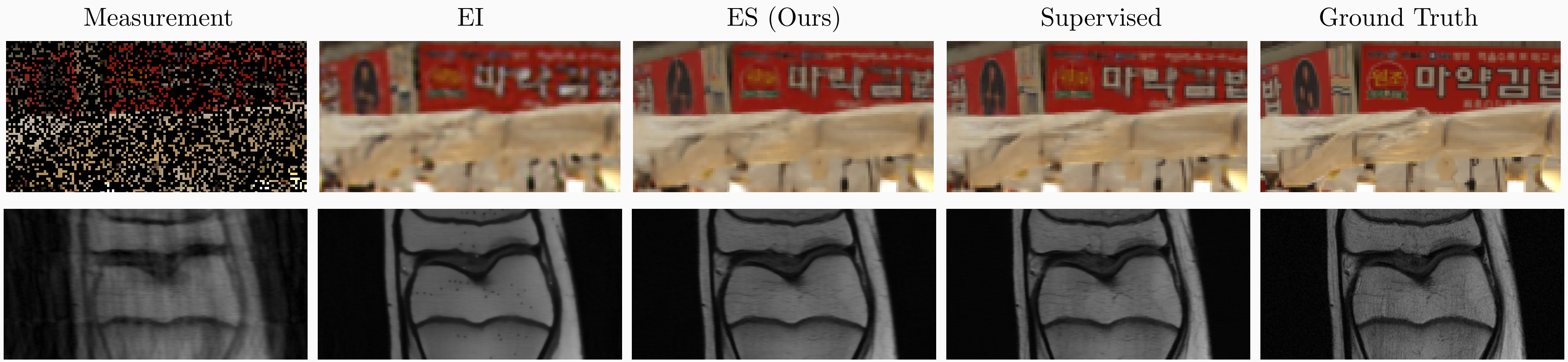
$$f(\boldsymbol{y}, \boldsymbol{A}) = \frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} \boldsymbol{T}_g r(\boldsymbol{y}, \boldsymbol{A}\boldsymbol{T}_g) \tag{7}$$

- **Artifact removal and unrolled networks** for an equivariant denoiser $\phi(\boldsymbol{x})$

$$\begin{cases} f(\boldsymbol{y}, \boldsymbol{A}) = \boldsymbol{x}_L \\ \boldsymbol{x}_{k+1} = \phi\left(\boldsymbol{x}_k - \gamma \nabla_{\boldsymbol{x}_k} d(\boldsymbol{A}\boldsymbol{x}_k, \boldsymbol{y})\right) \\ \boldsymbol{x}_0 = \boldsymbol{0} \end{cases} \tag{8}$$

- **MAP** and **MMSE** estimators for an invariant distribution $p(\boldsymbol{x})$

Reconstructions for inpainting and MRI



Conclusion

Equivariant Splitting

DeepInverse