Revenue Maximization Under Sequential Price Competition Via The Estimation Of s-Concave Demand Functions

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Problem Set-Up

Consider N sellers ($i = 1, 2, ..., N$), each selling a single type of product with unlimited inventories over a horizon of T periods. For $t = 1, 2, ..., T$ and for all $i \in \mathcal{N} = \{1, 2, ..., N\}$

1. **set price (made public)** $p_i^{(t)} \geq 0$ and observe $\mathbf{p}^{(t)} \triangleq (p_i^{(t)}, \mathbf{p}_{-i}^{(t)})$, where $\mathbf{p}_{-i}^{(t)} \triangleq (p_j^{(t)})_{j \in \mathcal{N} \setminus \{i\}}$ denote the competitor prices at time t.

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2. **observe demand (kept private)** $y_i^{(t)} = \lambda_i(\mathbf{p}^{(t)}) + \varepsilon_i^{(t)}$ where

$$\lambda_i(\mathbf{p}^{(t)}) \triangleq \psi_i(-\beta_i p_i^{(t)} + \gamma_i{}^\top \mathbf{p}_{-i}^{(t)}) = \psi_i(\theta_i{}^\top \mathbf{p}^{(t)}),$$

where $\varepsilon_i^{(t)}$ are iid (zero-mean) sub-gaussian noises and

$$\theta_i \triangleq (-\beta_i, \gamma_i) \in \mathbb{R}^N, \quad \|\theta_i\|_2 = 1, \quad \psi_i \text{ increasing}$$

Here $\varepsilon_i^{(t)}$ and $\varepsilon_j^{(t)}$ can be correlated with $i \neq j$, $i, j \in \mathcal{N}$.

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3. **get reward** $p_i^{(t)} y_i^{(t)}$.

Problem Set-Up (continued)

The objective of each learner is to minimize the following regret measure:

$$\mathrm{Reg}_i(T) = \mathbb{E} \sum_{t=1}^{T} \left[\mathrm{rev}_i(\Gamma_i(\mathbf{p}_{-i}^{(t)}) \mid \mathbf{p}_{-i}^{(t)}) - \mathrm{rev}_i(p_i^{(t)} \mid \mathbf{p}_{-i}^{(t)}) \right].$$

where

$$\mathrm{rev}_i(p_i \mid \mathbf{p}_{-i}^{(t)}) \triangleq \mathbb{E}[p_i y_i \mid p_i, \mathbf{p}^{(t)}] = p_i \psi_i(-\beta_i p_i + \boldsymbol{\gamma}_i^\top \mathbf{p}_{-i}^{(t)}),$$

and its maximum argument

$$\Gamma_i(\mathbf{p}_{-i}^{(t)}) \triangleq \mathrm{argmax}_{p_i}\ \mathrm{rev}_i(p_i \mid \mathbf{p}_{-i}^{(t)}), \quad i \in \mathcal{N}.$$

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Def: The Nash Equilibrium is the price vector $\mathbf{p}^*$ such that

$$\mathbf{p}^* = \boldsymbol{\Gamma}(\mathbf{p}^*) = (\Gamma_1(\mathbf{p}_{-i}^*), \ldots, \Gamma_N(\mathbf{p}_{-i}^*)).$$

Previous works

Find an algorithm that has

- **sublinear regret**: $\mathrm{Reg}_i(T) = O(T^{\eta})$, for some $\eta \in (0, 1)$.
- **converges to NE**: $\mathbb{E}(\|\mathbf{p}^{(t)} - \mathbf{p}^*\|_2) = O(T^{-\omega})$, for some $\omega \in (0, 1)$.

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Proposition: Existence of NE

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Proposition: Uniqueness of NE

If for every $i \in \mathcal{N}$, $s_i > -1/2$ and $\|\gamma_i\|_1 < \beta_i$ then a NE is unique.

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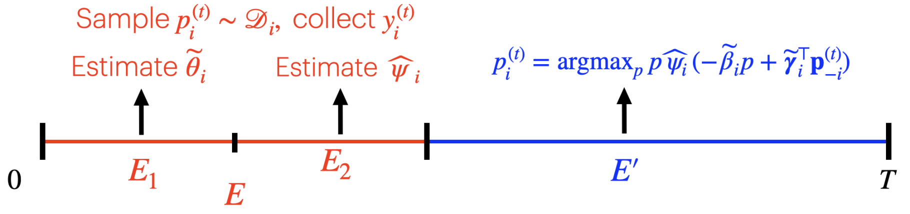

Theorem (informal) [Bracale D. et al.]

Under mild conditions on $\mathscr{D}_i$:

- For all $i \in \mathcal{N}$, $\|\theta_i - \widehat{\theta}_i\|_2 = O_P(|E_1|^{-1/2})$
- For all $i \in \mathcal{N}$, $\|\psi_i - \widehat{\psi}_i\|_\infty = O_P\left(\left(\frac{\log(|E_2|)}{|E_2|}\right)^{2/5}\right)$
- Under additional mild conditions on the revenue function, we have that for all $i \in \mathcal{N}$, $\text{Reg}_i(T) = O(T^{5/7})$ and $\mathbb{E}(\|\mathbf{p}^{(t)} - \mathbf{p}^*\|_2) = O(T^{-1/7})$

References I

[1] S. Li, C. Shi, and S. Mehrotra. LEGO: Optimal online learning under sequential price competition. *Major Revision at Operations Research. Available at SSRN 4803002*, 2024.