On the $O(1/T)$ Convergence of Alternating Gradient Descent-Ascent in Bilinear Games

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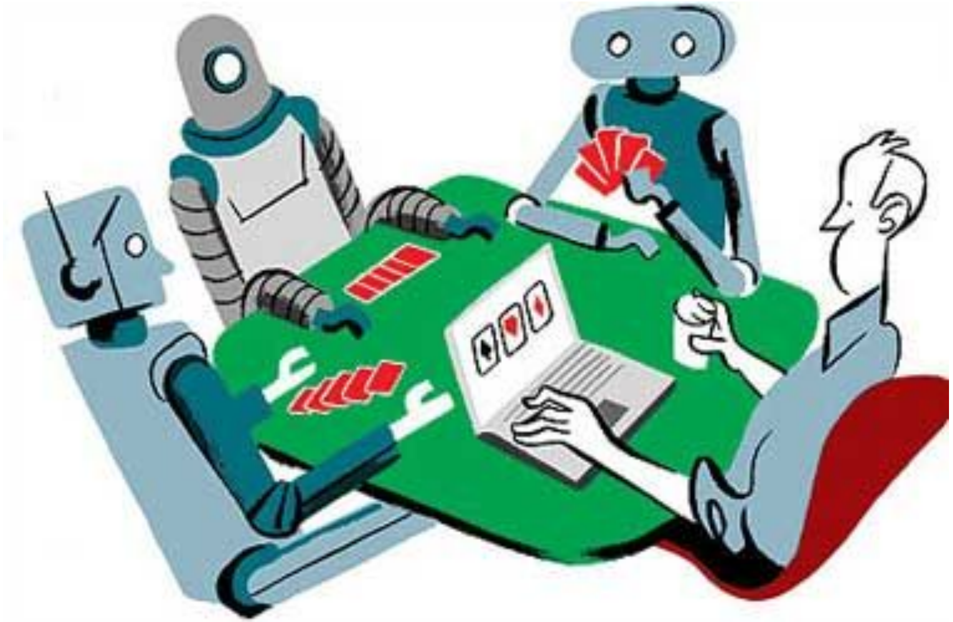


Applications:
Superhuman Poker AI

Bilinear Two-player Zero-sum Games

$$\min_{x \in \Delta_n} \max_{y \in \Delta_m} \overbrace{y^T A x}^{\text{Payoff: Bilinear}}$$

Decision Sets: Probability Simplex

Bilinear Two-player Zero-sum Games

$$\min_{x \in \Delta_n} \max_{y \in \Delta_m} \overbrace{y^T A}^{\text{Payoff gradient for x-player}} x$$

x-player: minimizer

Bilinear Two-player Zero-sum Games

y-player: maximizer

$$\min_{x \in \Delta_n} \max_{y \in \Delta_m} y^T \underbrace{Ax}_{\text{Payoff gradient for y-player}}$$

Nash Equilibrium (NE)

$$(y^*)^T A x^* = \min_{x \in \Delta_n} (y^*)^T A x$$

$$(y^*)^T A x^* = \max_{y \in \Delta_m} y^T A x^*$$

Gradient Descent-Ascent (GDA)

minimizer $x^{t+1} = \mathrm{Proj}_{\Delta n}(x^t - \eta A^T y^t)$

maximizer $y^{t+1} = \mathrm{Proj}_{\Delta m}(y^t + \eta A x^t)$

Alternating Gradient Descent-Ascent (AltGDA)

$$x^{t+1} = \mathrm{Proj}_{\Delta n}\left(x^t - \eta A^T y^t\right)$$

$$y^{t+1} = \mathrm{Proj}_{\Delta m}\left(y^t + \eta A x^{t+1}\right)$$

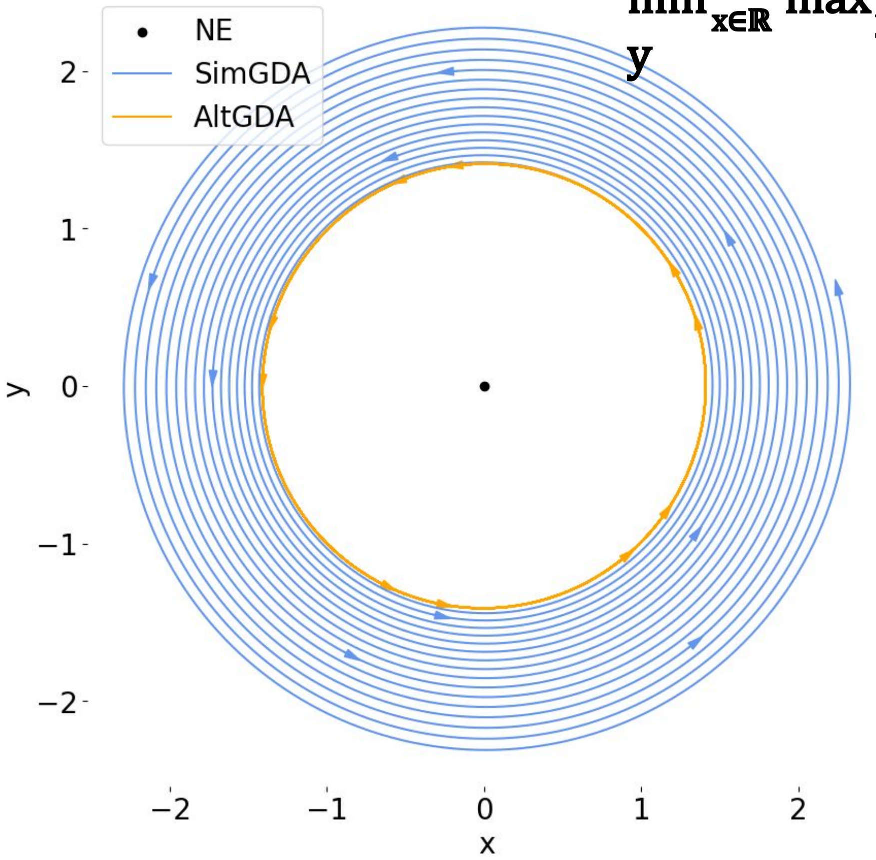
Simultaneous Gradient Descent-Ascent (SimGDA)

$$x^{t+1} = \mathrm{Proj}_{\Delta n}\left(x^t - \eta A^T y^t\right)$$

$$y^{t+1} = \mathrm{Proj}_{\Delta m}\left(y^t + \eta A x^t\right)$$

AltGDA vs. SimGDA: Trajectory Behaviors

$\min_{x\in\mathbb{R}} \max_{y\in\mathbb{R}} \; x y$




```python
SimGDA = []
x, y = 1, 1
SimGDA.append((x, y))
for _ in range(10000):
    x_, y_ = x, y
    x = x - 0.01 * y_
    y = y + 0.01 * x_
    SimGDA.append((x, y))
```

```python
AltGDA = []
x, y = 1, 1
AltGDA.append((x, y))
for _ in range(10000):
    x = x - 0.01 * y
    y = y + 0.01 * x
    AltGDA.append((x, y))
```

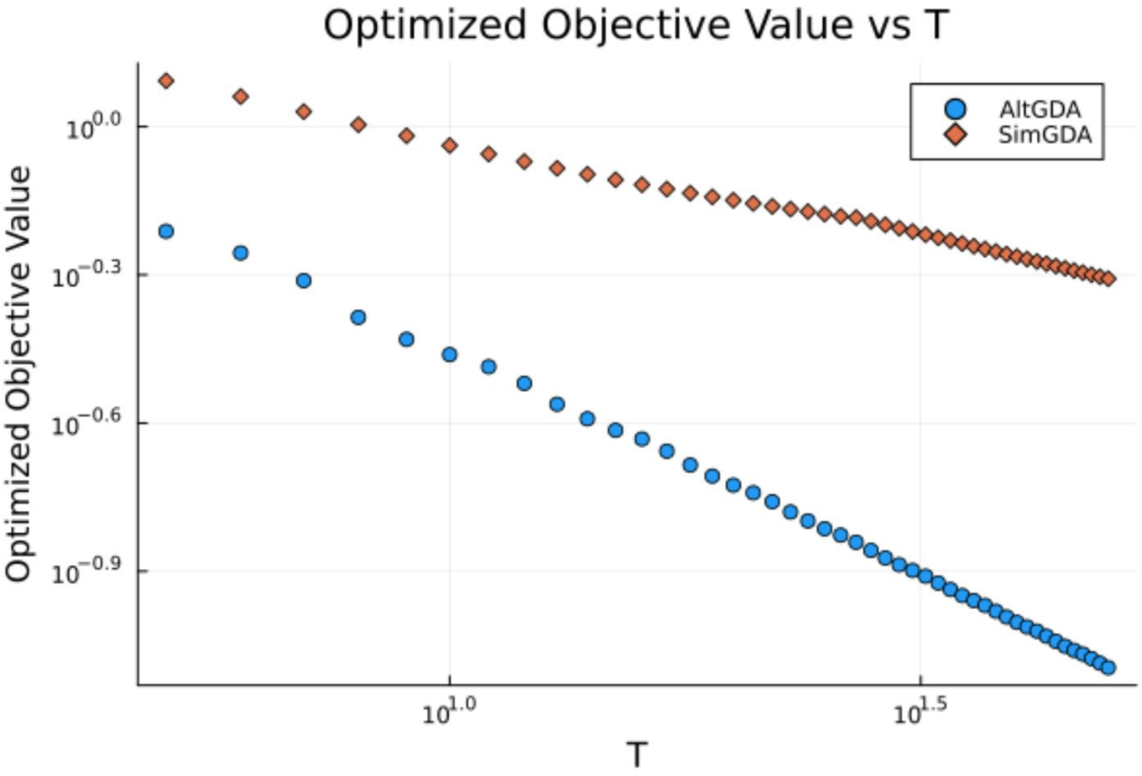
Theoretical Studies

- **Bailey, J.P., Gidel, G. and Piliouras, G. (2020)**: In the unconstrained minimax problems, AltGDA with a fixed stepsize converges at a rate of $\mathbf{O(1/T)}$
- Open Problem: Can we show a rate of $\mathbf{O(1/T)}$ for AltGDA in constrained setting?
- Surprising because $\mathbf{O(1/T)}$ convergence rate usually requires more complex techniques (e.g., optimism)

Performance Estimation Programming (PEP)

- Model worst-cases as solutions to optimization problems
- Those optimization problems are tractable for first-order methods in convex optimization
- In our case, for a fixed iteration budget T, by solving a semi-definite program (SDP), we can find the worst-case convergence rates over all possible instances

AltGDA vs. SimGDA: PEP Worst-case Analysis



Main Theoretical Results

Theorem 1. *Assume that the bilinear game admits an interior NE. Let* $\{(\boldsymbol{x}^t, \boldsymbol{y}^t)\}_{t=0,1,\ldots}$ *be a sequence of iterates generated by Algorithm 1 with* $\eta \leq \frac{1}{\|A\|_2} \min\{\min_{i\in[n]} x_i^*, \min_{j\in[m]} y_j^*\}$. *Then, we have*

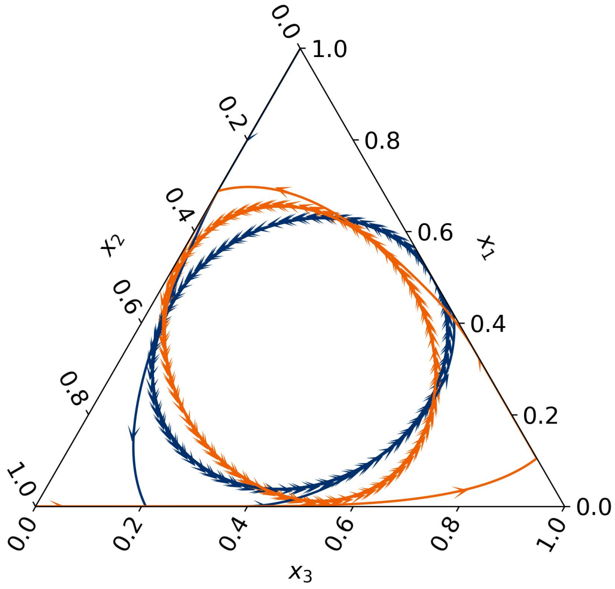
$$\texttt{DualityGap}\left(\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{x}^t, \frac{1}{T}\sum_{t=1}^{T}\boldsymbol{y}^t\right) \leq \frac{9 + 4\eta\|A\|_2}{\eta T}.$$

Theorem 2. *Let* $\{(\boldsymbol{x}^t, \boldsymbol{y}^t)\}_{t\geq 0}$ *be a sequence of iterates generated by Algorithm 1 with stepsize* $\eta \leq \frac{1}{2\|A\|_2}$ *and an initial point* $(\boldsymbol{x}^0, \boldsymbol{y}^0) \in S_0$, *where* S_0 *is defined in Eq. (7). Then, we have that*

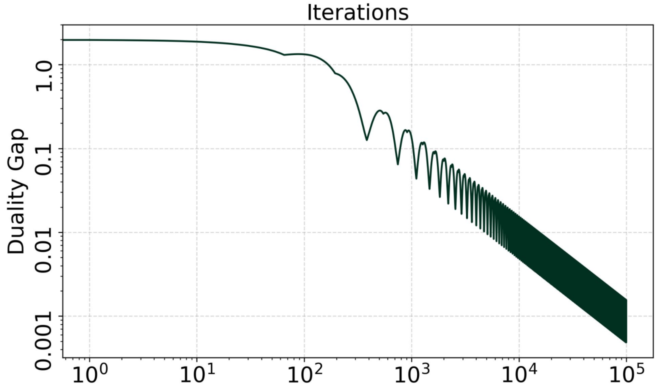
$$\texttt{DualityGap}\left(\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{x}^t, \frac{1}{T}\sum_{t=1}^{T}\boldsymbol{y}^t\right) \leq \frac{9 + 7\eta\|A\|_2 + (\delta^2/128)}{\eta T},$$

where δ *is defined in Eq. (6).*

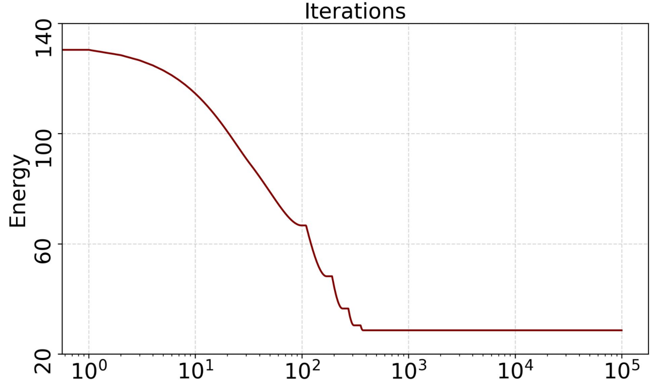
Key Observations



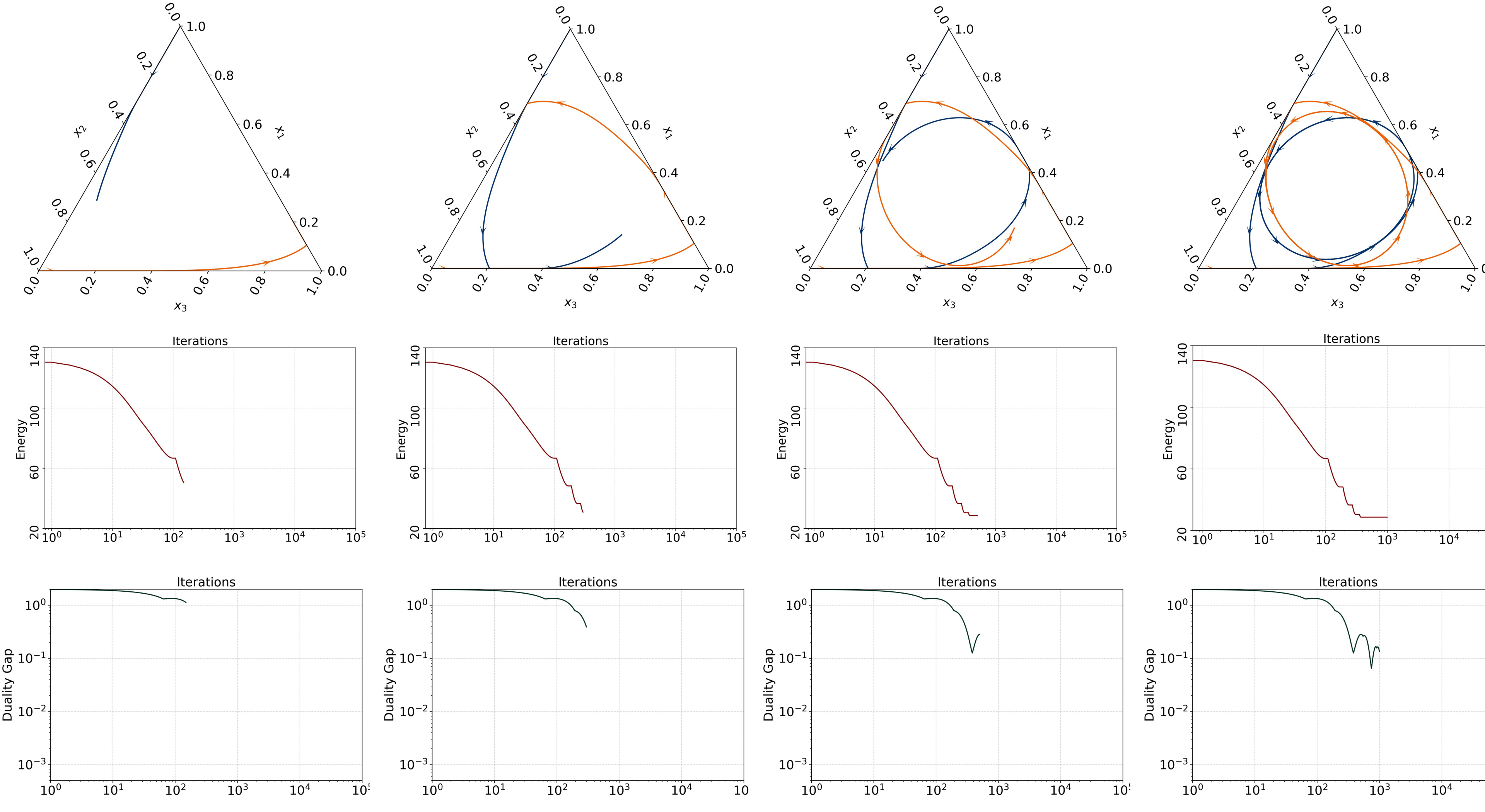
Trajectories



Convergence Rate

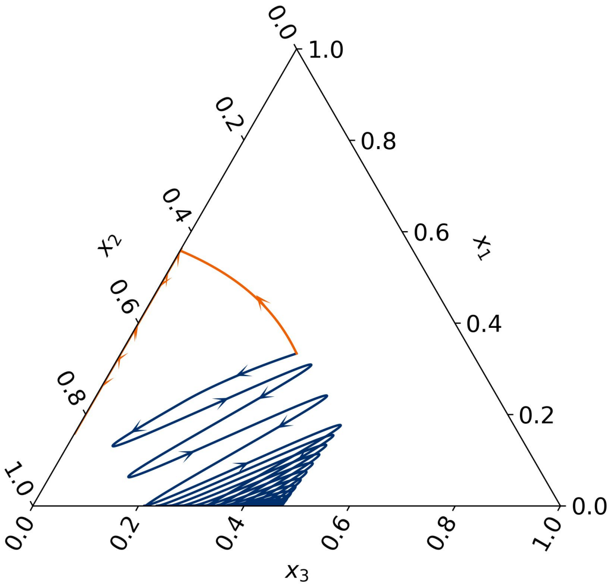


An Energy Function

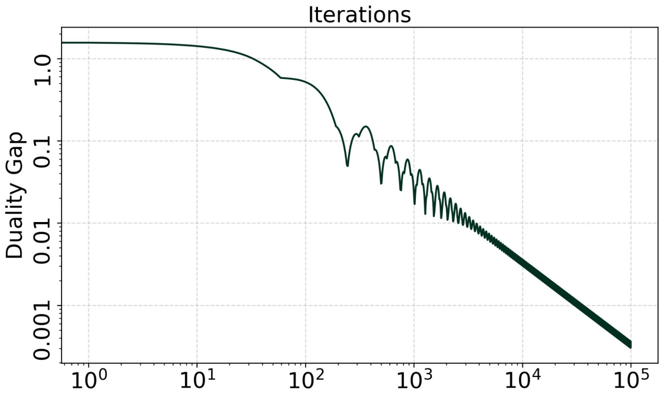


x_1
x_2
x_3
0.0
0.2
0.4
0.6
0.8
1.0
Iterations
Energy
140
100
60
20
10^0
10^1
10^2
10^3
10^4
10^5
Duality Gap
10^{-1}
10^{-2}
10^{-3}

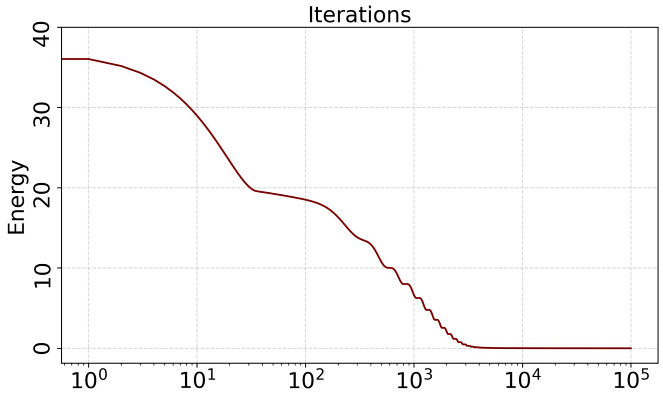
Key Observations



Trajectories



Duality Gap



An Energy Function

Thank you for listening!

Any discussion would be welcome!