Computing Equilibrium beyond Unilateral Deviation

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Minimum Average-Strong Equilibrium (MASE)

$$\pi^* \in \underset{\pi\in\Delta^{\mathcal{A}}}{\operatorname{argmin}} \max_{S\in\mathcal{S}} \max_{\hat{\boldsymbol{a}}_S\in\mathcal{A}_S} \frac{1}{|S|}\sum_{i\in S}\mathbb{E}_{\boldsymbol{a}\sim\pi}\left[\mathcal{U}_i\left(\hat{\boldsymbol{a}}_S, \boldsymbol{a}_{-S}\right) - \mathcal{U}_i\left(\boldsymbol{a}\right)\right].$$

A correlated strategy $\pi \in \Delta^{\mathcal{A}}$ is called an ϵ-*MASE* if

$$\max_{S\in\mathcal{S}} \max_{\hat{\boldsymbol{a}}_S\in\mathcal{A}_S} \frac{1}{|S|}\sum_{i\in S}\mathbb{E}_{\boldsymbol{a}\sim\pi}\left[\mathcal{U}_i\left(\hat{\boldsymbol{a}}_S, \boldsymbol{a}_{-S}\right) - \mathcal{U}_i\left(\boldsymbol{a}\right)\right]$$

$$\le \max_{S\in\mathcal{S}} \max_{\hat{\boldsymbol{a}}_S\in\mathcal{A}_S} \frac{1}{|S|}\sum_{i\in S}\mathbb{E}_{\boldsymbol{a}\sim\pi^*}\left[\mathcal{U}_i\left(\hat{\boldsymbol{a}}_S, \boldsymbol{a}_{-S}\right) - \mathcal{U}_i\left(\boldsymbol{a}\right)\right] + \epsilon.$$

Different Aggregation Functions

$$f(G) := \begin{cases} \frac{1}{|S|}\sum_{i\in S} G(i) & \text{Fixed Parameter Tractable ✓} \\ \sum_{i\in S} w_{S,i} G(i) & \text{Fixed Parameter Tractable ✓} \\ \max_{i\in S} G(i) & \text{Fixed Parameter Tractable ✓} \\ \min_{i\in S} G(i) & \text{NP-hard ✗} \end{cases}$$

Hardness Results

Theorem 4.1. Computing ϵ-MASE is NP-hard, even when $\mathcal{S}$ only contains singletons (coalitions of size one) and $1/\epsilon$ is polynomial in the number of players.

Definition 4.2 (Utility Dependency Graph). The utility dependence graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is an undirected graph with vertex set $\mathcal{V} = [N]$ representing the players, and edge set $\mathcal{E} = \bigcup_{k\in[N]}\{(i,j)\,|\,i,j\in\mathcal{N}(k), i\neq j\}$.

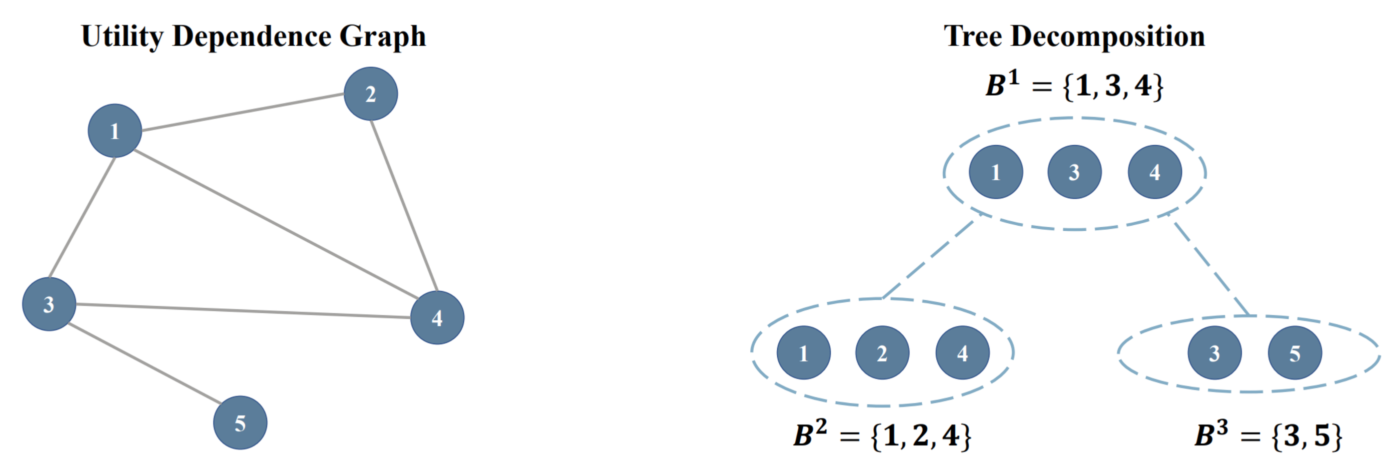


Figure 2: An illustration of a tree decomposition of the Utility Dependency Graph.

Theorem 4.3 (Treewidth). Suppose a tree decomposition of the Utility Dependency Graph is given. Under the Strong Exponential Time Hypothesis (SETH) (Impagliazzo & Paturi, 2001),[3] (MASE) cannot be computed in $\mathcal{O}^*((A-\zeta)^{\mathrm{tw}(\mathcal{G})})$ for any $\zeta > 0$. Moreover, under the additional assumption that BPP=P,[4] $\frac{1}{9N^2}$-approximate MASE cannot be computed in $\mathcal{O}^*((A-\zeta)^{\mathrm{tw}(\mathcal{G})})$ for any $\zeta > 0$.

Theorem 6.4. Approximating

$$\min_{\pi\in\Delta^{\mathcal{A}}} \max_{S\in\mathcal{S}} \max_{\hat{\pi}_S\in\Delta^{\mathcal{A}_S}} \min_{i\in S} \mathbb{E}_{\boldsymbol{a}\sim\pi,\hat{\boldsymbol{a}}_S\sim\hat{\pi}_S}\left[\mathcal{U}_i\left(\hat{\boldsymbol{a}}_S, \boldsymbol{a}_{-S}\right) - \mathcal{U}_i\left(\boldsymbol{a}\right)\right] \qquad \text{(Minimum Strong CCE)}$$

within an additive error of $\frac{1}{N^2}$ is NP-hard, even for a small game in which the joint action set size $|\mathcal{A}|$ is polynomial in the number of players N and the treewidth of the Utility Dependency Graph is two.

Efficient Computation of MASE

Theorem 7.1 (Efficient Representation). For any $\epsilon \ge 0$, at least one of the ϵ-MASE can be represented as a linear combination of $\sum_{S\in\mathcal{S}}|S|\cdot A^{\mathrm{tw}(\mathcal{G})}+1$ pure strategies, where $\mathrm{tw}(\mathcal{G})$ is the treewidth of Utility Dependency Graph.

Correlator-Deviator Meta Game

$$\min_{\pi\in\Delta^{\mathcal{A}}} \max_{\mu\in\Delta^{\times_{S\in\mathcal{S}}\mathcal{A}_S}} F(\pi,\mu), \tag{7.1}$$

where

$$F(\pi,\mu) := \sum_{S\in\mathcal{S}}\sum_{\hat{\boldsymbol{a}}_S\in\mathcal{A}_S}\frac{\mu(S,\hat{\boldsymbol{a}}_S)}{|S|}\sum_{i\in S}\mathbb{E}_{\boldsymbol{a}\sim\pi}\left[\mathcal{U}_i\left(\hat{\boldsymbol{a}}_S,\boldsymbol{a}_{-S}\right)-\mathcal{U}_i\left(\boldsymbol{a}\right)\right]. \tag{7.2}$$

Follow the Perturbed Leader

$$\pi^{(t+1)} \in \underset{\pi\in\Delta^{\mathcal{A}}}{\operatorname{argmin}} \sum_{\tau=1}^{t} F\left(\pi,\mu^{(\tau)}\right) - \left\langle\tilde{\boldsymbol{n}}^{(t+1)},\pi\right\rangle$$

$$\mu^{(t+1)} \in \underset{\mu^{(t)}\in\Delta^{\times_{S\in\mathcal{S}}\mathcal{A}_S}}{\operatorname{argmax}} \sum_{\tau=1}^{t} F\left(\pi^{(\tau)},\mu\right) + \left\langle\tilde{\boldsymbol{m}}^{(t+1)},\mu\right\rangle,$$

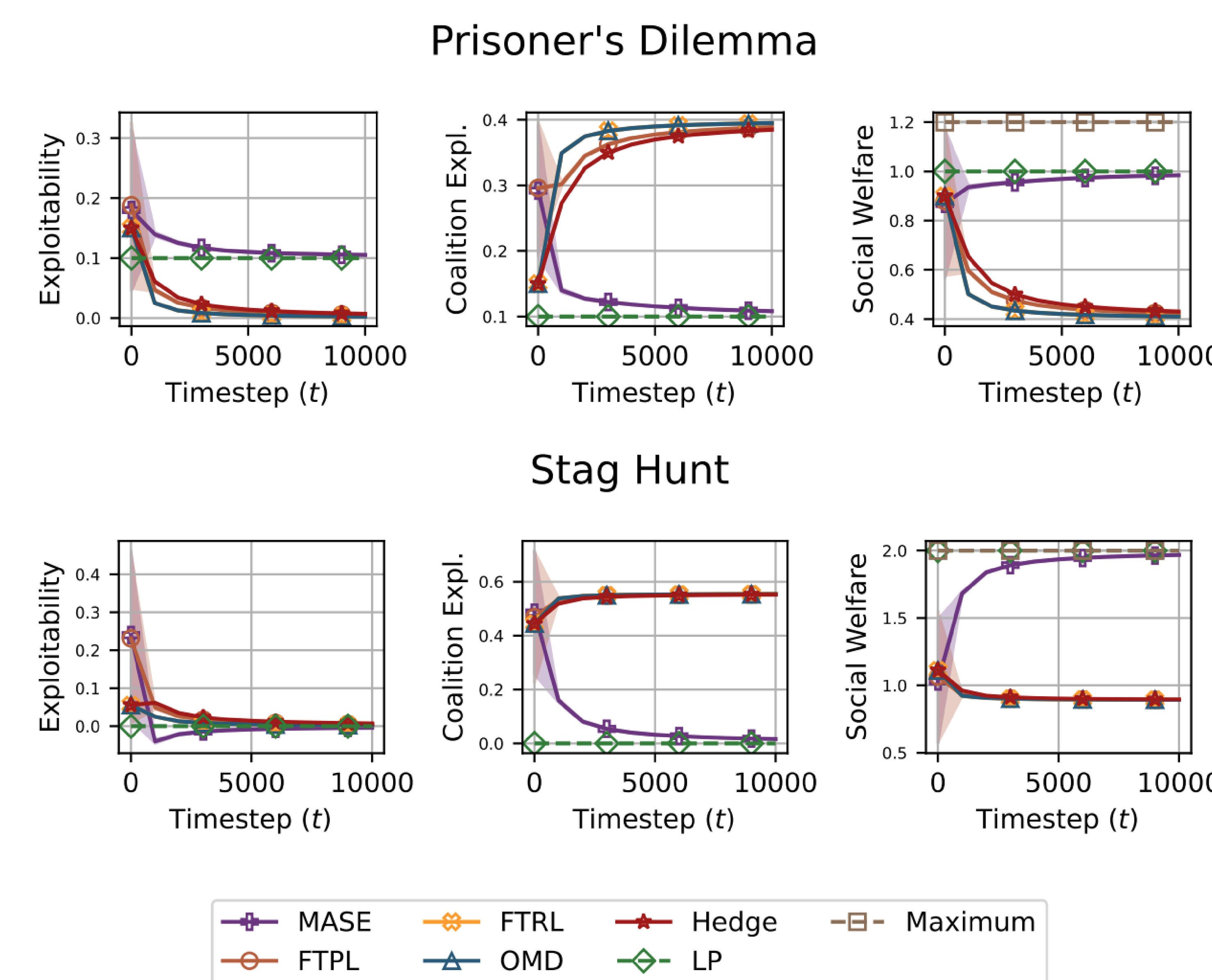
Tree decomposition. A tree decomposition $\mathcal{T} := B^1, B^2, \ldots, B^K$ of the Utility Dependency Graph $\mathcal{G} = (\mathcal{V},\mathcal{E})$ is a tree with K nodes (bags), each $B^k \subseteq \mathcal{V}$ where $\mathcal{V} = [N]$, satisfying the following properties (Diestel, 2025):

1. $\bigcup_{k=1}^K B^k = [N]$.
2. For every edge $(i,j)\in\mathcal{E}$, there exists k with $\{i,j\}\subseteq B^k$.
3. For any player $i\in[N]$, if i appears in two bags $B, B'\in\mathcal{T}$, then every bag on the path from B to B' also contains i.

Theorem 7.4. Let π^*, μ^* be the solution of (7.1), and define $\bar{\pi} := \frac{1}{T}\sum_{t=1}^T \pi^{(t)}$, $\bar{\mu} := \frac{1}{T}\sum_{t=1}^T \mu^{(t)}$. Then, for any $\delta > 0$, with probability at least $1-\delta$, we have

$$\max_{\hat{\mu}\in\Delta^{\times_{S\in\mathcal{S}}\mathcal{A}_S}} F(\bar{\pi},\hat{\mu}) \le F(\pi^*,\mu^*) + 4|\mathcal{T}|\frac{1+(\mathrm{tw}(\mathcal{G})+1)\log A}{\eta T} + 4\eta|\mathcal{T}| + 2\sqrt{\frac{2\log\frac{2}{\delta}}{T}}.$$

Experiments



Coalition Exploitability